NORMANHURST BOYS HIGH SCHOOL

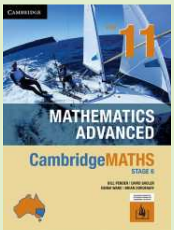
MATHEMATICS ADVANCED

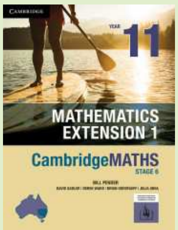
YEAR 11 COURSE

Topic summary and exercises:

With references to

Ⓐ **Algebric Techniques & Equations**





Name: ....

Initial version by H. Lam, 2018 (Algebraic Techniques), 2016 (Coordinate Geometry). Last updated January 11, 2026.
Based on the work from the legacy syllabuses by R. Trenwith, 1995–2010, subsequently maintained by H. Lam, 2011-18.
Additional editing by I. Ham and M. Ho, 2019-2020, A. Sun in late 2020, and H. Lam again for 2026 new syllabuses
Various corrections by students & members of the Mathematics Departments at North Sydney Boys and Normanhurst Boys High Schools.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at http://www.flaticon.com, used under CC BY 2.0.

Symbols used

Beware! Heed warning.

Ⓐ Mathematics Advanced content, for `MAA` coded class

(X1) Mathematics Extension 1 content, for `MAX` coded class

Literacy: note new word/phrase.

Facts/formulae to memorise.

On the course Reference Sheet.

ICT usage

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Content outcomes addressed

MAV-11-01 Applies algebraic techniques and the laws of indices and surds to manipulate expressions and solve problems

MAV-11-02 Uses functions and relations to model, analyse and solve problems

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Further exercises from
  - Ⓐ *Cambridge Year 11 2 Unit* (Pender, Sadler, Shea, & Ward, 2009)
  - Ⓐ *Cambridge Year 11 Advanced* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019a)
  - (X1) *Cambridge Year 11 3 Unit* (Pender, Sadler, Shea, & Ward, 1999)
  - (X1) *Cambridge Year 11 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019b)

  will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Learning intentions & outcomes

✓ Content

☐ 1.1 Algebraic techniques

- Use index laws to simplify expressions and solve problems involving positive, negative, zero or fractional indices
- Expand, factorise and simplify algebraic expressions
- Simplify expressions involving algebraic fractions
- Expand and simplify expressions involving surds
- Identify the conjugate of $\sqrt{a} \pm \sqrt{b}$, and rationalise the denominators of expressions of the form $\frac{a}{\sqrt{b} \pm \sqrt{c}}$ and $\frac{\sqrt{a} \pm \sqrt{d}}{\sqrt{b} \pm \sqrt{c}}$, where a, b, c and d are positive rational numbers
- Solve quadratic equations $ax^2 + bx + c = 0$ by factorisation, completing the square and using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where a, b and c are real numbers and $a \neq 0$
- Define the discriminant as $\Delta = b^2 - 4ac$ and use it to solve problems involving the number of real roots of a quadratic equation and determine the conditions for roots to be equal, distinct, real or rational

✓ Content

☐ 1.2 Linear functions

- Determine the equations of straight lines in gradient–intercept form $y = mx + c$ with gradient m and y-intercept
- Determine the equations of straight lines in general form $ax + by + c = 0$, where a, b and c are constants
- Determine the equation of a straight line passing through a point (x_1, y_1) with gradient m using the point–gradient formula $y - y_1 = m(x - x_1)$.
- Determine the equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) by calculating its gradient m using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$
- Examine and use the relationship between the angle of inclination of a line or tangent to a curve, θ, with the positive x-axis, and the gradient, m, of that line or tangent, and establish that $\tan\theta = m$
- Determine the x-intercept and y-intercept of a straight line given its equation
- Choose and apply appropriate techniques to graph a straight line given its equation
- Find the equation of a line that is parallel or perpendicular to a given line
- Solve linear inequalities and graph the solution on a number line

---

☐ 1.3 Constructing and using functions

- Construct and use linear functions to model and solve problems in real-world situations, identifying the independent and dependent variables and any restrictions on these variables, and justify conclusions in the context of the problem
- Use linear inequalities to model and solve problems in real-world situations, and justify conclusions in the context of the problem
- Solve practical problems involving a pair of simultaneous linear equations both algebraically and graphically, with and without graphing applications, and justify conclusions in the context of the problem
- Construct and use simultaneous equations to model and solve a problem where cost and revenue are represented by linear equations, identify and analyse the break-even point, and justify conclusions in the context of the problem

---

Contents

Section 1

Indices and surd laws

Learning Goal(s)

Knowledge
Index laws

Skills
Simplifying index laws

Understanding
The difference between *simplifying* and *solving*

By the end of this section am I able to:

1.1 Algebraic techniques

- Use index laws to simplify expressions and solve problems involving positive, negative, zero or fractional indices
- Expand and simplify expressions involving surds
- Identify the conjugate of $\sqrt{a} \pm \sqrt{b}$, and rationalise the denominators of expressions of the form $\frac{a}{\sqrt{b} \pm \sqrt{c}}$ and $\frac{\sqrt{a} \pm \sqrt{d}}{\sqrt{b} \pm \sqrt{c}}$, where a, b, c and d are positive rational numbers

1.1 Index laws

1.1.1 Operations resulting in addition/subtraction/multiplication of indices

Only the most challenging questions are provided here. The rest are omitted for brevity and should be assumed knowledge.

Laws/Results

For $a > 1$:

$$\text{(Product)} \qquad a^x \times a^y = a^{x+y} \tag{1.1}$$

$$\text{(Quotient)} \qquad a^x \div a^y = a^{x-y} \tag{1.2}$$

$$\text{(Power)} \qquad (a^x)^y = a^{x-y} \tag{1.3}$$

$$\text{(Power 0)} \qquad a^0 = 1 \tag{1.4}$$

$$\text{(Negative index)} \qquad a^{-x} = \frac{1}{a^x} \tag{1.5}$$

$$\text{(Fractional index)} \qquad a^{\frac{1}{x}} = \sqrt[x]{a} \tag{1.6}$$

Laws/Results

(A slightly more neglected rule)

$$(ab)^x = \underline{a^x b^x} \tag{1.7}$$

Important note

Use these laws in either "direction", i.e. from L to R *or* R to L

Example 1

[Ex 8A Q15(e)] Write as a single fraction, without negative indices and simplify:

$$x^{-2}y^{-2}\left(x^2y^{-1} - y^2x^{-1}\right)$$

Definition 1

Simplify write the expression with less strokes .
(This is an informal definition)

Example 2

Fully simplify: $\dfrac{2^{n+1} - 2^n}{2^{2n+1} - 2^{2n}}$

Answer: 2^{-n}

Example 3

[**Ex 8A Q16(f)**] Fully simplify:

$$\frac{24^{x+1} \times 8^{-1}}{6^{2x}}$$

Answer: $2^x 3^{1-x}$

Example 4

[**Ex 8A Q18(c)**] Fully simplify: $\dfrac{12^n - 18^n}{3^n - 2^n}$.

Answer: -6^n

1.1.2 Fractional indices

Example 5

Fully simplify: $\sqrt[3]{\dfrac{x^{5k+1}y^{4k-5}}{x^{2(k-1)}y^{-2k+1}}}$

Answer: $x^{k+1}y^{2(k-1)}$

1.1.3 Simple index law equations

Example 6

[**Ex 8B Q18**] By converting to the same base, solve for x:

$$64^x = \sqrt{32}$$

Further exercises

(A) **Ex 7A** (Pender et al., 2019a)

- Q10-22

(A) **Ex 7B** (Pender et al., 2019a)

- Q5-7, 12-13, 17-18

(X1) **Ex 8A** (Pender et al., 2019b)

- Q10-18

(X1) **Ex 8B** (Pender et al., 2019b)

- Q4-6, 13-16

1.2 Basic arithmetic of surds

Theorem 1

Fully simplify surds by finding their ……greatest…… ……perfect…… ……square…… factor.

Example 7

Fully simplify: $\sqrt{72} - \sqrt{50} + \sqrt{12}$ **Answer:** $\sqrt{2} + 2\sqrt{3}$

Example 8

Expand and fully simplify: $\left(\sqrt{15} - \sqrt{6}\right)^2$ **Answer:** $21 - 6\sqrt{10}$

Theorem 2

Rewrite surds with a ......rational...... denominator by multiplying numerator and denominator by the ......conjugate...... of the denominator.

Example 9

Fully simplify: $\dfrac{1}{2\sqrt{3}-3\sqrt{2}}$.

Answer: $-\frac{2\sqrt{3}+3\sqrt{2}}{6}$

1.3 Equality of surds

Theorem 3

Two surds $a + b\sqrt{c}$ and $x + y\sqrt{c}$ are equal iff ..... $a = x$ ..... and ..... $b = y$ ..... .

Example 10

Find the value of a and b if $\dfrac{2}{\sqrt{5}+1} = a + b\sqrt{5}$.

Answer: $a = -\frac{1}{2}, b = \frac{1}{2}$

Additional questions

Acknowledgement: Portions taken from Grove (2010, Ex 2.23)

1. Find the values of a and b if

(a) $\dfrac{3}{2\sqrt{5}} = \dfrac{\sqrt{a}}{b}$

(b) $\dfrac{\sqrt{3}}{4\sqrt{2}} = \dfrac{a\sqrt{6}}{b}$

(c) $\dfrac{2\sqrt{7}}{\sqrt{7}-4} = a + b\sqrt{7}$

(d) $\dfrac{\sqrt{2}+3}{\sqrt{2}-1} = a + \sqrt{b}$

2. Show that $\dfrac{\sqrt{2}-1}{\sqrt{2}+1} + \dfrac{4}{\sqrt{2}}$ is rational.

3. If $x = \sqrt{3}+2$, fully simplify:

(a) $x + \dfrac{1}{x}$

(b) $\left(x + \dfrac{1}{x}\right)^2$

(c) $x^2 + \dfrac{1}{x^2}$

4. If $2 + \dfrac{1}{x} = \sqrt{3}$ where $x \neq 0$, find the exact value of x (with a rational denominator).

Answers

1. (a) $a = 45, b = 10$ (b) $a = 1, b = 8$ (c) $a = -\frac{14}{9}, b = -\frac{8}{9}$ (d) $a = 5, b = 32$ **2.** Show **3.** (a) 4 (b) 16 (c) 14 **4.** $-\left(\sqrt{3}+2\right)$

Further exercises

Ⓐ **Ex 2C** (Pender et al., 2019a)
- Q1-8 last column

Ⓐ **Ex 2D** (Pender et al., 2019a)
- Q1-11 last 2 columns

Ⓐ **Ex 2E** (Pender et al., 2019a)
- last 2 columns

(X1) **Ex 2C** (Pender et al., 2019b)
- Q1-8 last column

(X1) **Ex 2D** (Pender et al., 2019b)
- Q1-13 last 2 columns

(X1) **Ex 2E** (Pender et al., 2019b)
- last 2 columns

Section 2

Expansions, factorisations and algebraic fractions

2.1 Expansions and simplification

Learning Goal(s)

Knowledge
Stage 5 algebra

Skills
Expand and factorise algebraic expressions

Understanding
The need to be able to perform expansions or factorisations to further simplify expressions

By the end of this section am I able to:

1.1 Algebraic techniques

- Expand, factorise and simplify algebraic expressions
- Simplify expressions involving algebraic fractions

Only the most challenging questions are provided here. The rest are omitted for brevity and should be assumed knowledge.

Laws/Results

Square of a sum/difference

$$(a \pm b)^2 = a^2 + 2ab + b^2$$

Difference of squares

$$(a-b)(a+b) = a^2 - b^2$$

Example 11

Expand and fully simplify $(a+b+c)^2$ by rewriting as $\Big(a+(b+c)\Big)^2$.

Answer: $a^2+b^2+c^2+2ab+2ac+2bc$

2.2 Factorisations

Simply, to rewrite a sum of two or more terms into a .... multiplication .... of two or more terms.

Laws/Results

Four elementary methods of factorisation, which can be nested within each other:

Highest common factor

$$5a^2b + 10ab^2 = 5ab(a+2b)$$

Difference of squares

$$a^2 - b^2 = (a-b)(a+b)$$

Quadratic trinomial

$$a^2 - 4a - 77 = (a-11)(a+7)$$

Grouping in pairs

$$\begin{aligned} 3x - 3y + ay - ax &= x(3-a) - y(3-a) \\ &= (x-y)(3-a) \end{aligned}$$

Gentle reminder

For further revision, see the *A complete course on factorisations*, located at Lowe and Lam (2010)

Additional questions

1. Simplify

(a) $2a - 3b - a + 2b$ (b) $5x^2 - 3x + x - x^2 + 2x$ (c) $a - (0.1)a$

2. Simplify

(a) $3a^4 \times 2a$

(b) $5a^3 \times 3a^2b^2$

(c) $(-3x) \times (-4x)$

(d) $(3a^2)^3$

(e) $(-2a^2b)^5$

(f) $16x^8 \div 2x^2$

(g) $\dfrac{6x^3y^2z}{9x^2y^4z}$

(h) $(pq^3)^2 \times p^3q \div (pq)^3$

(i) $x(2x+1)^3 \times x^4(2x+1)$

(j) $t \times \dfrac{1}{t}$

(k) $\sqrt{x} \times \sqrt{x}$

(l) reciprocal of $a + b$

3. (a) From $3x - 2y + 1$ take $x - y + 1$.

(b) What must be added to $2m^2 - m + 1$ to give $5m^2 - 3m$?

(c) How many times must p be subtracted form p^3 to give zero?

(d) What is the product of x and

i. half the reciprocal of x? ii. the reciprocal of half x?

4. Expand, simplifying where necessary:

(a) $3(2a - 5)$

(b) $3ab(2a + 5b)$

(c) $-(1 - x)$

(d) $(2x - 1)(x + 3)$

(e) $(3a - 2)^2$

(f) $(3q^2 + 2)^2$

(g) $(1 - 2x)(1 + 2x)$

(h) $(w^3 + 2v^2)(w^3 - 2v^2)$

(i) $2(a - 3) - (1 - 2a)$

(j) $1 - (x - 1)^2$

(k) $(a+b)^2 - (a-b)(a+b)$

(l) $3(v - 2)^2$

(m) $\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$

(n) $a(a + 2)(3a - 4)$

(o) $(a + b)(a - 2b + 1)$

(p) $(a - b)(a^2 + ab + b^2)$

(q) $(x - 2)(x^2 + 3x - 1)$

(r) $(a - 1)(a - 2)^2$

(s) $(a - 2)^3$

(t) $(2a + 1)^4$

(u) $\left(x + \dfrac{1}{x}\right)^2$

5. (a) Find an expansion for $(a + b + c)^2$ by first rewriting it as $\left[(a + b) + c\right]^2$. Try to guess the expansion for $(a + b + c + d)^2$.

(b) A square has side length a. The length of the square is increased by b where $0 < b < a$, while the width decreased by b. Does the area increase or decrease? By how much?

(c) In a right angled triangle, the length of the hypotenuse is $(x + 4)$ cm and the length of one of the short sides is $(x - 4)$ cm. Find an expression, in simplest form, for the length of the other short side.

6. Factorise *fully*:

(a) $6a^2b^3 - 9ab^5$
(b) $a(b+1) - b(b+1)$
(c) $9a^2 - 16b^2$
(d) $-4x + 6$
(e) $x^2 - 3x - 54$
(f) $3x^2 - 8x - 3$
(g) $12x^2 + x - 6$
(h) $x^3 + x^2 + x + 1$
(i) $x^3 + x^2 - x - 1$
(j) $x^4 - 1$
(k) $4x^2 - 16y^2$
(l) $x^2 - 6x + 9$
(m) $4x^2 + 12xy + 9y^2$
(n) $\ell^4 - 4mn\ell^2 + 4m^2n^2$
(o) $6a^2 + 7ab - 3b^2$
(p) $x^2 - 2\frac{1}{4}$
(q) $\frac{a^2}{9} - \frac{4}{25}$
(r) $(a+b)^2 - c^2$
(s) $a^2 - (b+c)^2$
(t) $(a+b)^2 - (c-d)^2$
(u) $x^2 - y^2 - x - y$
(v) $x^3 + 6x^2 + 9x$
(w) $-x^2 + 2x + 15$
(x) $x^4 - x^2 - 12$
(y) $a(a-b)^2 - ac^2$
(z) $\left(x^2 - y^2\right)^2 - (x-y)^4$

7. Show that $(x+1)^2 - (x-1)^2 = 4x$ by

(a) expanding and simplifying, and
(b) factorising, without expanding.

8. Expand and factorise $1 - x\Big(1 - x(1-x)\Big)$.

9. Factorise fully (without expanding):

(a) $3(x+1)^2 + 2x(x+1)$
(b) $x^4 - x^2(2x-1)$
(c) $x^2(x-1)^3 - 2x(x-1)^2$

10. If $f(n) = n(n+1)(n+2)$, express $f(n) + f(n+1)$ in factored form.

Further exercises

Ⓐ **Ex 1B** Pender et al. (2019a)
- Q3-7, 9

Ⓐ **Ex 1C** Pender et al. (2019a)
- Q1-7

ⓍⅠ **Ex 1B** Pender et al. (2019b)
- Q7, 8, 9, 10

ⓍⅠ **Ex 1C** Pender et al. (2019b)
- Q1-9

Answers

1. (a) $a-b$ (b) $4x^2$ (c) $\frac{9a}{10}$ or $0.9a$ **2.** (a) $6a^5$ (b) $15a^5b^2$ (c) $12x^2$ (d) $27a^6$ (e) $-32a^{10}b^5$ (f) $8x^6$ (g) $\frac{2x}{3y^2}$ (h) p^2q^4 (i) $x^5(2x+1)^4$ (j) 1 (k) x (l) $\frac{1}{a+b}$ note: NOT $\frac{1}{a}+\frac{1}{b}$ **3.** (a) $2x-y$ (b) $3m^2-2m-1$ (c) p^2 times (d) i. $\frac{1}{2}$ ii. 2 **4.** (a) $6a-15$ (b) $6a^2b+15ab^2$ (c) $x-1$ (d) $2x^2+5x-3$ (e) $9a^2-12a+4$ (f) $9q^4+12q^2+4$ (g) $1-4x^2$ (h) w^6-4v^4 (i) $4a-7$ (j) $2x-x^2$ (k) $2b^2+2ab$ (l) $3v^2-12v+12$ (m) x^2-3 (n) $3a^3+2a^2-8a$ (o) $a^2-ab+a+b-2b^2$ (p) a^3-b^3 (q) x^3+x^2-7x+2 (r) a^3-5a^2+8a-4 (s) $a^3-6a^2+12a-8$ (t) $16a^4+32a^3+24a^2+8a+1$ (u) $x^2+2+\frac{1}{x^2}$ **5.** (a) $a^2+b^2+c^2+2ab+2ac+2bc$; $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$. (b) decreases by b^2 (c) $4\sqrt{x}$ cm **6.** (a) $3ab^3\left(2a-3b^2\right)$ (b) $(b+1)(a-b)$ (c) $(3a-4b)(3a+4b)$ (d) $-2(2x-3)$ (e) $(x-9)(x+6)$ (f) $(3x+1)(x-3)$ (g) $(4x+3)(3x-2)$ (h) $(x+1)(x^2+1)$ (i) $(x+1)^2(x-1)$ (j) $(x-1)(x+1)(x^2+1)$ (k) $4(x-2y)(x+2y)$ (l) $(x-3)^2$ (m) $(2x+3y)^2$ (n) $\left(\ell^2-2mn\right)^2$ (o) $(3a-b)(2a+3b)$ (p) $\left(x-\frac{3}{2}\right)\left(x+\frac{3}{2}\right)$ (q) $\left(\frac{a}{3}-\frac{2}{5}\right)\left(\frac{a}{3}+\frac{2}{5}\right)$ (r) $(a+b+c)(a+b-c)$ (s) $(a+b+c)(a-b-c)$ (t) $(a+b+c-d)(a+b-c+d)$ (u) $(x+y)(x-y-1)$ (v) $x(x+3)^2$ (w) $-(x-5)(x+3)$ (x) $(x-2)(x+2)(x^2+3)$ (y) $a(a-b+c)(a-b-c)$ (z) $4xy(x-y)^2$ **7.** "Show" question. **8.** $(1-x)\left(1+x^2\right)$ **9.** (a) $(x+1)(5x+3)$ (b) $x^2\left(x-1\right))^2$ (c) $x(x-1)^2(x+1)(x-2)$ **10.** $(n+1)(n+2)(2n+3)$

2.3 Algebraic Fractions

Important note

- Obtain a ....common.......... ....denominator.......... prior to adding.
- Only multiply up by as much as required.

Example 12

Fully simplify:

(a) $\dfrac{1}{x-4}-\dfrac{1}{x}$

(b) $\dfrac{2}{x^2-x}-\dfrac{5}{x^2-1}$

Answer: (a) $\frac{4}{x(x-4)}$ (b) $\frac{2-3x}{x(x-1)(x+1)}$

Additional questions

1. Simplify fully. Also state any values of pronumeral(s) for which the simplification is not valid.

(a) $\dfrac{4a+6}{4}$ (d) $\dfrac{x^2-2x-3}{x+1}$ (g) $\dfrac{4a-8}{12-6a}$ (j) $\dfrac{1-b^2}{b^3-1}$

(b) $\dfrac{4a+6}{6a+9}$ (e) $\dfrac{x+1}{x^2-2x-3}$ (h) $\dfrac{a^2-4}{a+2}$ (k) $\dfrac{x^4-5x^2+4}{x^2-x-2}$

(c) $\dfrac{x^2}{5x-x^2}$ (f) $\dfrac{1-x}{x-1}$ (i) $\dfrac{a^2+a-6}{a^2-3a+2}$ (l) $\dfrac{x^3+2x^2-4x-8}{x^2+4x+4}$

2. Write as a single fraction in simplest form.

(a) $\dfrac{a}{2}+\dfrac{2a}{3}$ (e) $\dfrac{4}{a}-\dfrac{3}{b}$ (i) $\dfrac{3}{a^3b}-\dfrac{2}{a^2b^4}$

(b) $\dfrac{m-1}{2}-\dfrac{2m-3}{5}$ (f) $\dfrac{1}{a^2}+\dfrac{2}{a}$ (j) $\dfrac{2}{x-2}+\dfrac{3}{x+1}$

(c) $x+\dfrac{1}{x}$ (g) $\dfrac{a+1}{2a}-\dfrac{a-2}{3a}$ (k) $\dfrac{2x}{x+3}-\dfrac{x-2}{x+1}$

(d) $a-\dfrac{a+b}{3}$ (h) $\dfrac{1}{(x-3)^2}+\dfrac{1}{x-3}$ (l) $\dfrac{1}{x^2+x}+\dfrac{1}{x^2-x}$

(m) $\dfrac{x}{(x-2)(x+2)}+\dfrac{x+1}{(x-2)(x-1)}$ (p) $\dfrac{1}{x^2-x-2}-\dfrac{1}{x^2+5x+4}-\dfrac{1}{x^2+2x-8}$

(n) $\dfrac{2}{y+2}-\dfrac{1}{y+3}+\dfrac{5}{y-1}$ (q) $\dfrac{a^2}{a^2+3a+2}-\dfrac{2a}{a+2}$

(o) $\dfrac{k}{k+1}+\dfrac{1}{k^2+3k+2}$

3. Simplify fully:

(a) $\dfrac{4x^2}{3y^2}\times\dfrac{6y}{15x^4}$ (f) $\dfrac{9}{x^3+64}\div\dfrac{6}{x+4}$

(b) $\dfrac{3a^3}{7b^2}\div\dfrac{9a}{7b}$ (g) $\dfrac{x+y}{a-b}\times\dfrac{b-a}{y+x}$

(c) $\dfrac{2}{a}\div a$ (h) $\dfrac{a^2-a}{6a^3+6a^2}\div\dfrac{a^2-1}{8a}$

(d) $\dfrac{x+1}{2}\times\dfrac{4x}{(x+1)^2}$ (i) $\dfrac{a^2-2a-3}{a^2+3a}\times\dfrac{3a^2+18a+27}{a^2-9}$

(e) $\dfrac{1}{x+2}\times\dfrac{4x+8}{3}$

4. x is the smallest of three consecutive integers.

(a) Find as a single fraction in simplest form, an expression for the sum of the reciprocals of these integers.

(b) Three fractions are formed by dividing each of these integers by the integer following it. Find an expression, in simplest form, for the product of these fractions.

5. Find the reciprocal of $\dfrac{1}{a}+\dfrac{1}{b}$.

6. Simplify:

(a) $\dfrac{\frac{1}{m}+\frac{1}{n}}{m+n}$ (b) $\dfrac{1}{1-\frac{m}{n}}+\dfrac{1}{1-\frac{n}{m}}$ (c) $\dfrac{1-\frac{2}{t+1}}{t-\frac{2}{t+1}}$

7. (a) If $f(n)=n(n+1)(n+2)$, simplify $\dfrac{f(n)}{f(n+1)}$.

(b) If $f(n)=\dfrac{n^2}{n-1}$, prove that $f\left(\dfrac{t}{t-1}\right)=f(t)$.

Answers

1. (a) $\frac{2a+3}{2}$ (b) $\frac{2}{3}$ $\left[a\neq-\frac{3}{2}\right]$ (c) $\frac{x}{5-x}$ $[x\neq0,5]$ (d) $x-3$ $[x\neq1]$ (e) $\frac{1}{x-3}$ $[x\neq-1,3]$ (f) -1 $[x\neq1]$ (g) $-\frac{2}{3}$ $[a\neq2]$ (h) $a-2$ $[a\neq-2]$ (i) $\frac{a+3}{a-1}$ $[a\neq1,2]$ (j) $-\frac{b+1}{b^2+b+1}$ $[b\neq1]$ (k) $(x+2)(x-1)$ $[x\neq-1,2]$ (l) $x-2$ $[x\neq-2]$ **2.** (a) $\frac{a}{6}$ (b) $\frac{m+1}{10}$ (c) $\frac{x^2+1}{x}$ (d) $\frac{2a-b}{3}$ (e) $\frac{4b-3a}{ab}$ (f) $\frac{2a+1}{a^2}$ (g) $\frac{a+7}{6a}$ (h) $\frac{x-2}{(x-3)^2}$ (i) $\frac{3b^3-2a}{a^3b^4}$ (j) $\frac{5x-4}{(x-2)(x+1)}$ (k) $\frac{x^2+x+6}{(x+3)(x+1)}$ (l) $\frac{2}{x^2-1}$ (m) $\frac{2(x^2+x+1)}{(x-2)(x+1)(x+2)}$ (n) $\frac{2(3y^2+14y+13)}{(y+2)(y+3)(y-1)}$ (o) $\frac{k+1}{k+2}$ (p) $\frac{-x+5}{(x-2)(x+1)(x+4)}$ (q) $-\frac{a}{a+1}$ **3.** (a) $\frac{8}{15x^2y}$ (b) $\frac{a^2}{3b}$ (c) $\frac{2}{a^2}$ (d) $\frac{2x}{x+1}$ (e) $\frac{4}{3}$ (f) $\frac{3}{2(x^2-4x+16)}$ (g) -1 (h) $\frac{4}{3(a+1)^2}$ (i) $\frac{3(a+1)}{a}$ **4.** (a) $\frac{3x^2+6x+2}{x(x+1)(x+2)}$ (b) $\frac{x}{x+3}$ **5.** $\frac{ab}{a+b}$ (NOT $a+b$!!!) **6.** (a) $\frac{1}{mn}$ (b) 1 (c) $\frac{1}{t+2}$ **7.** (a) $\frac{n}{n+3}$

Further exercises

Ⓐ **Ex 1D** (Pender et al., 2019a)

- Q1-11 last column

Ⓧ1 **Ex 1D** (Pender et al., 2019b)

- Q2-13 last column

Section 3

Linear functions

3.1 Gradient of lines

Learning Goal(s)

Knowledge
Various forms of equations of straight lines

Skills
Use coordinate geometry methods to solve problems

Understanding
Relationship between coordinate geometry methods and equation solving

By the end of this section am I able to:

1.2 Linear functions

- Determine the equations of straight lines in gradient–intercept form $y = mx + c$ with gradient m and y-intercept
- Determine the equations of straight lines in general form $ax + by + c = 0$, where a, b and c are constants
- Determine the equation of a straight line passing through a point (x_1, y_1) with gradient m using the point–gradient formula $y - y_1 = m(x - x_1)$.
- Determine the equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) by calculating its gradient m using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$
- Examine and use the relationship between the angle of inclination of a line or tangent to a curve, θ, with the positive x-axis, and the gradient, m, of that line or tangent, and establish that $\tan\theta = m$
- Determine the x-intercept and y-intercept of a straight line given its equation
- Choose and apply appropriate techniques to graph a straight line given its equation
- Find the equation of a line that is parallel or perpendicular to a given line

Laws/Results

The gradient formula to calculate the gradient between two points (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_2}{x_2 - x_1}$$

Derivation of formula:

Definition 2

The angle of inclination θ of the line and the positive direction of the x axis:

$$m = \tan\theta$$

Derivation of formula:

Laws/Results

Parallel lines have the same gradient.

Laws/Results

The gradient of two perpendicular lines are negative reciprocals:

$$m_1 = -\frac{1}{m_2} \quad \text{or} \quad m_1 m_2 = -1$$

Definition 3

Three or more points are **collinear** (pronounded *co-linear*) if a single straight line passes through all points.

Definition 4

Three or more lines are **concurrent** if all lines pass through the same point.

Example 13

Show that the points $A(-2,5)$, $B(1,3)$ and $C(7,-1)$ are collinear

Example 14

[Ex 5B Q16] Answer the following for the points $W(2,3)$, $X(-7,5)$, $Y(-1,-3)$ and $Z(5,-1)$:

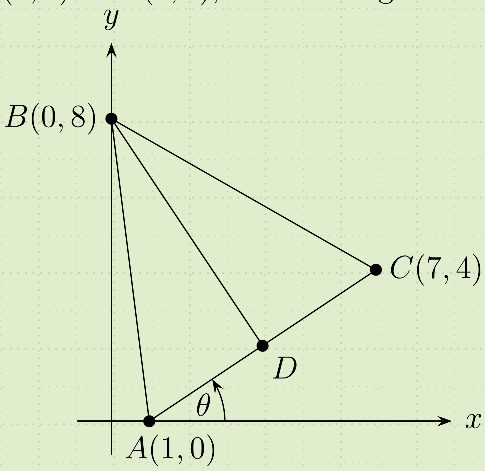
(a) Show that WZ is parallel to XY.

(b) Find the lengths WZ and XY. Hence deduce the type of quadrilateral $WXYZ$.

(c) Show that that the diagonals WY and XZ are perpendicular.

Example 15

[Ex 5D Q11] (Pender et al., 1999) The points A, B and C have coordinates $(1,0)$, $(0,8)$ and $(7,4)$, and the angle between AC and the x axis is θ.



(a) Find the gradient of the line AC and hence determine θ to the nearest degree. **2**

(b) Find the equation of AC. **2**

(c) Find the coordinates of D, the midpoint of AC. **2**

(d) Show that $AC \perp BD$. **1**

(e) What type of triangle is ABC? Show full reasoning. **2**

(f) Find the area of this triangle. **2**

(g) Write down the coordinates of the point E such that $ABCE$ is a rhombus. **2**

Further exercises

Ⓐ **Ex 6B** (Pender et al., 2009)

- Q7-21 odd #

(X1) **Ex 5B** (Pender et al., 1999)

- Q7-21 odd #

Exercise 5B

NOTE: Diagrams should be drawn wherever possible.

1. Find the gradient of a line (i) parallel to, (ii) perpendicular to a line with gradient:

(a) 2 (b) -1 (c) $\dfrac{3}{4}$ (d) $-\dfrac{p}{q}$

2. Find the gradient of each interval AB, then find the gradient of a line perpendicular to it:

(a) $(1,4)$, $(5,0)$
(b) $(-2,-7)$, $(3,3)$
(c) $(-5,-2)$, $(3,2)$
(d) $(3,6)$, $(5,5)$
(e) $(-1,-2)$, $(1,4)$
(f) $(-a,b)$, $(3a,-b)$

3. Find the gradient, to two decimal places, of a line with angle of inclination:

(a) 15° (b) 135° (c) $22\frac{1}{2}^\circ$ (d) 72°

4. (a) What angle of inclination (to the nearest degree where necessary) does a line with each gradient make with the x-axis? Does the line slope upwards or downwards?

(i) 1 (ii) $-\sqrt{3}$ (iii) 4 (iv) $\dfrac{1}{\sqrt{3}}$

(b) Find the acute angle made by each line in part (a) with the y-axis.

5. Find the points A and B where each line meets the x-axis and y-axis respectively. Hence find the gradient of AB and its angle of inclination α (to the nearest degree):

(a) $y = 3x + 6$
(b) $y = -\frac{1}{2}x + 1$
(c) $3x + 4y + 12 = 0$
(d) $\dfrac{x}{3} - \dfrac{y}{2} = 1$
(e) $4x - 5y - 20 = 0$
(f) $\dfrac{x}{2} + \dfrac{y}{5} = 1$

6. Given $A = (2,3)$, write down the coordinates of any three points P such that AP has gradient 2.

DEVELOPMENT

7. Find the gradient, to two decimal places, of a line sloping upwards if its acute angle with the y-axis is:

(a) 15° (b) 45° (c) $22\frac{1}{2}^\circ$ (d) 72°

8. Find the gradients of PQ and QR, and hence determine whether P, Q and R are collinear:

(a) $P(-2,7)$, $Q(1,1)$, $R(4,-6)$ (b) $P(-5,-4)$, $Q(-2,-2)$, $R(1,0)$

9. (a) Triangle ABC has vertices $A(-1,0)$, $B(3,2)$ and $C(4,0)$. Calculate the gradient of each side and hence show that $\triangle ABC$ is a right-angled triangle.

(b) Do likewise for the triangles with the vertices given below. Then find the lengths of the sides enclosing the right angle, and calculate the area of each triangle:

(i) $P(2,-1)$, $Q(3,3)$, $R(-1,4)$ (ii) $X(-1,-3)$, $Y(2,4)$, $Z(-3,2)$

10. Quadrilateral $ABCD$ has vertices $A(-1,1)$, $B(3,-1)$, $C(5,3)$ and $D(1,5)$.

(a) Show that it has two pairs of parallel sides. (b) Confirm that $AB \perp BC$.
(c) Show also that $AB = BC$. (d) What type of quadrilateral is $ABCD$?

11. Use gradients to show that each quadrilateral $ABCD$ below is a parallelogram. Then show that it is:

(a) a rhombus, for the vertices $A(2,1)$, $B(-1,3)$, $C(1,0)$ and $D(4,-2)$,
(b) a rectangle, for the vertices $A(4,0)$, $B(-2,3)$, $C(-3,1)$ and $D(3,-2)$,
(c) a square, for the vertices $A(3,3)$, $B(-1,2)$, $C(0,-2)$ and $D(4,-1)$.

12. The interval PQ has gradient -3. A second line passes through $A(-2,4)$ and $B(1,y)$. Find the value of y if: (a) AB is parallel to PQ, (b) AB is perpendicular to PQ.

13. Find λ for the points $X(-1,0)$, $Y(1,\lambda)$ and $Z(\lambda,2)$, if $\angle YXZ = 90^\circ$.

14. For the four points $P(k,1)$, $Q(-2,-3)$, $R(2,3)$ and $S(1,k)$, it is known that PQ is parallel to RS. Find the possible values of k.

15. On a number plane, mark the origin O and the points $A(2,1)$ and $B(3,-1)$.
(a) Find the gradients of OA and AB and hence show that they are perpendicular.
(b) Show that $OA = AB$. (c) Find the midpoint D of OB.
(d) Given that D is the midpoint of AC, find the coordinates of C.
(e) What shape best describes quadrilateral $OABC$?

16. Answer the following questions for the points $W(2,3)$, $X(-7,5)$, $Y(-1,-3)$ and $Z(5,-1)$.
(a) Show that WZ is parallel to XY.
(b) Find the lengths WZ and XY. Hence deduce the type of the quadrilateral $WXYZ$.
(c) Show that the diagonals WY and XZ are perpendicular.

17. Quadrilateral $ABCD$ has vertices $A(1,-4)$, $B(3,2)$, $C(-5,6)$ and $D(-1,-2)$.
(a) Find the midpoints P of AB, Q of BC, R of CD, and S of DA.
(b) Prove that $PQRS$ is a parallelogram by showing that $PQ \parallel RS$ and $PS \parallel QR$.

19. Given the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$, find and simplify the gradient of PQ.

20. The points $A(4,-2)$, $B(-4,4)$ and $P(x,y)$ form a right angle at P. Form an equation in x and y, and hence find the equation of the curve on which P lies. Describe this curve.

21. The points $O(0,0)$, $P(4,0)$ and $Q(x,y)$ form a right angle at Q and $PQ = 1$ unit.
(a) Form a pair of equations for x and y.
(b) Solve them simultaneously to find the coordinates of the two possible locations of Q.

Exercise 5B (Page 165)

1(a) $2, -\frac{1}{2}$ **(b)** $-1, 1$ **(c)** $\frac{3}{4}, -\frac{4}{3}$ **(d)** $-\frac{p}{q}, \frac{q}{p}$
2(a) $-1, 1$ **(b)** $2, -\frac{1}{2}$ **(c)** $\frac{1}{2}, -2$ **(d)** $-\frac{1}{2}, 2$
(e) $3, -\frac{1}{3}$ **(f)** $-\frac{b}{2a}, \frac{2a}{b}$
3(a) $0{\cdot}27$ **(b)** $-1{\cdot}00$ **(c)** $0{\cdot}41$ **(d)** $3{\cdot}08$
4(a)(i) 45°, upwards **(ii)** 120°, downwards
(iii) 76°, upwards **(iv)** 30°, upwards
(b)(i) 45° **(ii)** 30° **(iii)** 14° **(iv)** 60°
5(a) $m = 3$, $\alpha \doteqdot 72^\circ$ **(b)** $m = -\frac{1}{2}$, $\alpha \doteqdot 153^\circ$
(c) $m = -\frac{3}{4}$, $\alpha \doteqdot 143^\circ$ **(d)** $m = \frac{2}{3}$, $\alpha \doteqdot 34^\circ$
(e) $m = \frac{4}{5}$, $\alpha \doteqdot 39^\circ$ **(f)** $m = -\frac{5}{2}$, $\alpha \doteqdot 112^\circ$
6 Check your answers by substitution into the gradient formula.
7(a) $3{\cdot}73$ **(b)** $1{\cdot}00$ **(c)** $2{\cdot}41$ **(d)** $0{\cdot}32$
8(a) non-collinear **(b)** collinear with gradient $\frac{2}{3}$
9(a) $m_{AB} = \frac{1}{2}$, $m_{BC} = -2$ and $m_{AC} = 0$, so $AB \perp BC$.
(b)(i) $m_{PQ} = 4$, $m_{QR} = -\frac{1}{4}$ and $m_{PR} = -\frac{5}{3}$, so $PQ \perp QR$. Area $= 8\frac{1}{2}$ units^2
(ii) $m_{XY} = \frac{7}{3}$, $m_{YZ} = \frac{2}{5}$ and $m_{XZ} = -\frac{5}{2}$, so $XZ \perp YZ$. Area $= 14\frac{1}{2}$ units^2
10(d) square
11 In each case, show that each pair of opposite sides is parallel. **(a)** Show also that two adjacent sides are equal. **(b)** Show also that two adjacent sides are perpendicular. **(c)** Show that it is both a rhombus and a rectangle.
12(a) -5 **(b)** 5
13 $\lambda = -\frac{1}{2}$
14 $k = 2$ or -1
15(a) OA has gradient $\frac{1}{2}$, and OB has gradient 2. Their product is $\frac{1}{2} \times (-2) = -1$, thus they are perpendicular. **(b)** $OA = AB = \sqrt{5}$ **(c)** $D(\frac{3}{2}, -\frac{1}{2})$
(d) $C(1,-2)$ **(e)** square
16(b) $WZ = 5$, $XY = 10$. It is a trapezium, but not a parallelogram.
17(a) $P = (2,-1)$, $Q = (-1,4)$, $R = (-3,2)$, $S = (0,-3)$
(b) $m_{PQ} = m_{RS} = -\frac{5}{3}$ and $m_{PS} = m_{QR} = 1$
18(a) $P = (2,3)$, $Q = (3,5)$
(b) $m_{PQ} = m_{BC} = 2$ and $PQ = \sqrt{5}$
19 $\frac{1}{2}(p+q)$
20 $x^2 + (y-1)^2 = 5^2$, a circle with centre $(0,1)$ and radius 5.
21(a) $\dfrac{y^2}{x(x-4)} = -1$ (products of gradients is -1), and $(x-4)^2 + y^2 = 1$
(b) $Q(\frac{15}{4}, \frac{1}{4}\sqrt{15})$ or $Q(\frac{15}{4}, -\frac{1}{4}\sqrt{15})$
23(b) They are collinear if and only if $\Delta = 0$, that is $a_1b_2 + a_2b_3 + a_3b_1 = a_2b_1 + a_3b_2 + a_1b_3$.
24(a) $x = \frac{4p}{1-p^2}$ **(b)** $x = p - \frac{1}{p}$

3.2 Equations of lines

Definition 5

The **gradient-intercept form** of the equation of a line:

$$y = mx + b$$

where

- m – gradient
- b – y intercept

Definition 6

The **general form** of the equation of a line:

$$ax + by + c = 0$$

Laws/Results

The **point-gradient formula** to find the equation of a straight line through (x_1, y_1) with a given gradient m:

$$\frac{y - y_1}{x - x_1} = m$$

Derivation of formula:

Important note

No new theory is in this section. However, expect slightly more difficult problems within textbook exercises.

Example 16

[**Ex 5D Q6**] Given the points $A(1, -2)$ and $B(-3, 4)$, find in general form the equation of:

(a) the line AB,

(b) the line through A perpendicular to AB.

Example 17

[Ex 5D Q12]

(a) On a number plane, plot the points $A(4,3)$, $B(0,-3)$ and $C(4,0)$.

(b) Find the equation of BC.

(c) Explain why $OACB$ is a parallelogram.

(d) Find the area of $OACB$ and the length of the diagonal AB.

Further exercises

A **Ex 6C** (Pender et al., 2009)

- Q5, 10-16

A **Ex 6D** (Pender et al., 2009)

- Q5
- Q7 last 3 columns
- Q14, 18-20

X1 **Ex 5C** (Pender et al., 1999)

- Q6, 10-14

X1 **Ex 5D** (Pender et al., 1999)

- Q4, 7-10, 13-18

Exercise 5C

1. Determine by substitution whether the point $A(3,-2)$ lies on the line:
(a) $y = 4x - 10$ (b) $8x + 10y - 4 = 0$ (c) $x = 3$

2. Write down the coordinates of any three points on the line $2x + 3y = 4$.

3. Write down the equations of the vertical and horizontal lines through:
(a) $(1,2)$ (b) $(-1,1)$ (c) $(3,-4)$ (d) $(5,1)$ (e) $(-2,-3)$ (f) $(-4,1)$

4. Write down the gradient and y-intercept of each line:
(a) $y = 4x - 2$ (b) $y = \frac{1}{5}x - 3$ (c) $y = 2 - x$

5. Use the formula $y = mx + b$ to write down the equation of each of the lines specified below, then put that equation into general form:
(a) with gradient 1 and y-intercept 3
(b) with gradient -2 and y-intercept 5
(c) with gradient $\frac{1}{5}$ and y-intercept -1
(d) with gradient $-\frac{1}{2}$ and y-intercept 3

6. Solve each equation for y and hence write down the gradient and y-intercept:
(a) $x - y + 3 = 0$
(b) $y + x - 2 = 0$
(c) $2x - y = 5$
(d) $x - 3y + 6 = 0$
(e) $3x + 4y = 5$
(f) $2y - 3x = -4$

7. For each line in question 6, substitute $y = 0$ and $x = 0$ to find the points A and B where the line intersects the x-axis and y-axis respectively, and hence sketch the curve.

8. For each line in question 6, use the formula gradient $= \tan\alpha$ to find its angle of inclination, to the nearest minute where appropriate.

9. Show by substitution that the line $y = mx + b$ passes through $A(0,b)$ and $B(1, m+b)$. Then show that the gradient of AB, and hence of the line, is m.

DEVELOPMENT

10. Find the gradient of each line below, and hence find in gradient–intercept form the equation of a line passing through $A(0,3)$ and (i) parallel to it, (ii) perpendicular to it:
(a) $2x + y + 3 = 0$ (b) $5x - 2y - 1 = 0$ (c) $3x + 4y - 5 = 0$

11. In each part below, the angle of inclination α and the y-intercept A of a line are given. Use the formula gradient $= \tan\alpha$ to find the gradient of each line, then find its equation in general form:
(a) $\alpha = 45°$, $A = (0,3)$
(b) $\alpha = 60°$, $A = (0,-1)$
(c) $\alpha = 30°$, $A = (0,-2)$
(d) $\alpha = 135°$, $A = (0,1)$

12. A triangle is formed by the x-axis and the lines $5y = 9x$ and $5y + 9x = 45$. Find (to the nearest degree) the angles of inclination of the two lines, and hence show that the triangle is isosceles.

13. Consider the two lines ℓ_1: $3x - y + 4 = 0$ and ℓ_2: $x + ky + \ell = 0$. Find the value of k if:
(a) ℓ_1 is parallel to ℓ_2, (b) ℓ_1 is perpendicular to ℓ_2.

Exercise **5C** (Page 169)

1(a) not on the line **(b)** on the line **(c)** on the line
2 Check your answer by substitution.
3(a) $x = 1,\ y = 2$ **(b)** $x = -1,\ y = 1$
(c) $x = 3,\ y = -4$ **(d)** $x = 5,\ y = 1$
(e) $x = -2,\ y = -3$ **(f)** $x = -4,\ y = 1$
4(a) $m = 4,\ b = -2$ **(b)** $m = \frac{1}{5},\ b = -3$
(c) $m = -1,\ b = 2$
5(a) $x - y + 3 = 0$ **(b)** $2x + y - 5 = 0$
(c) $x - 5y - 5 = 0$ **(d)** $x + 2y - 6 = 0$
6(a) $m = 1,\ b = 3$ **(b)** $m = -1,\ b = 2$
(c) $m = 2,\ b = -5$ **(d)** $m = \frac{1}{3},\ b = 2$
(e) $m = -\frac{3}{4},\ b = \frac{5}{4}$ **(f)** $m = 1\frac{1}{2},\ b = -2$
7 The sketches are clear from the intercepts.
(a) $A(-3, 0),\ B(0, 3)$ **(b)** $A(2, 0),\ B(0, 2)$
(c) $A(2\frac{1}{2}, 0),\ B(0, -5)$ **(d)** $A(-6, 0),\ B(0, 2)$
(e) $A(1\frac{2}{3}, 0),\ B(0, 1\frac{1}{4})$ **(f)** $A(1\frac{1}{3}, 0),\ B(0, -2)$
8(a) $\alpha = 45^\circ$ **(b)** $\alpha = 135^\circ$ **(c)** $\alpha \doteqdot 63^\circ 26'$
(d) $\alpha \doteqdot 18^\circ 26'$ **(e)** $\alpha \doteqdot 143^\circ 8'$ **(f)** $\alpha \doteqdot 56^\circ 19'$
10(a) $y = -2x + 3,\ y = \frac{1}{2}x + 3$
(b) $y = \frac{5}{2}x + 3,\ y = -\frac{2}{5}x + 3$
(c) $y = -\frac{3}{4}x + 3,\ y = \frac{4}{3}x + 3$
11(a) $x - y + 3 = 0$ **(b)** $-\sqrt{3}x + y + 1 = 0$
(c) $x - \sqrt{3}y - 2\sqrt{3} = 0$ **(d)** $x + y - 1 = 0$
12 The angles of inclination are about 61° and 119°, so the two lines make acute angles of 61° with the x-axis.
13(a) $k = -\frac{1}{3}$ **(b)** $k = 3$
14(a) $2x - y = -4$ **(b)** $x - y = -3$ **(c)** $5x + y = -3$
15 $(x - a)^2 + (y - x)^2 = a^2$,
where $a = 2 - \sqrt{2}$ or $a = 2 + \sqrt{2}$,
$(x - \sqrt{2}\,)^2 + (y + \sqrt{2}\,)^2 = 2$,
$(x + \sqrt{2}\,)^2 + (y - \sqrt{2}\,)^2 = 2$
16(a) From their gradients, two pairs of lines are parallel and two lines are perpendicular.
(b) The distance between the x-intercepts of one pair of lines must equal the distance between the y-intercepts of the other pair. Thus $k = 2$ or 4.

Exercise 5D

NOTE: Selection of questions from this long exercise will depend on students' previous knowledge.

1. Use point–gradient form $y - y_1 = m(x - x_1)$ to find in general form the equation of the line:

(a) through $(1, 1)$ with gradient 2,
(b) with gradient -1 through $(3, 1)$,
(c) with gradient 3 through $(-5, -7)$,
(d) through $(0, 0)$ with gradient -5,
(e) through $(-1, 3)$ with gradient $-\frac{1}{3}$,
(f) with gradient $-\frac{4}{5}$ through $(3, -4)$.

2. Find the gradient of the line through each pair of given points, and hence find its equation:

(a) $(3, 4)$, $(5, 8)$ (b) $(-1, 3)$, $(1, -1)$ (c) $(-4, -1)$, $(6, -6)$ (d) $(5, 6)$, $(-1, 4)$

3. Write down the equation of the line with the given intercepts, then rewrite it in general form:

(a) $(-1, 0)$, $(0, 2)$ (b) $(2, 0)$, $(0, 3)$ (c) $(0, -1)$, $(-4, 0)$ (d) $(0, -3)$, $(3, 0)$

4. (a) Find the point M of intersection of the lines ℓ_1: $x + y = 2$ and ℓ_2: $4x - y = 13$.
(b) Show that M lies on ℓ_3: $2x - 5y = 11$, and hence that ℓ_1, ℓ_2 and ℓ_3 are concurrent.
(c) Use the same method to test whether each set of lines is concurrent:
(i) $2x + y = -1$, $x - 2y = -18$ and $x + 3y = 15$
(ii) $6x - y = 26$, $5x - 4y = 9$ and $x + y = 9$

5. Find the gradient of each line below and hence find, in gradient–intercept form, the equation of a line: (i) parallel to it passing through $A(3, -1)$, (ii) perpendicular to it passing through $B(-2, 5)$.

(a) $2x + y + 3 = 0$ (b) $5x - 2y - 1 = 0$ (c) $4x + 3y - 5 = 0$

6. Given the points $A(1, -2)$ and $B(-3, 4)$, find in general form the equation of:

(a) the line AB, (b) the line through A perpendicular to AB.

DEVELOPMENT

7. The angle of inclination α and a point A on a line are given below. Use the formula gradient $= \tan\alpha$ to find the gradient of each line, then find its equation in general form:

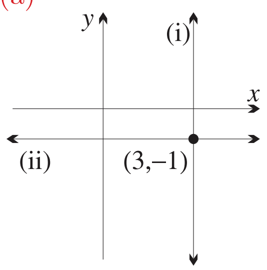
(a) $\alpha = 45°$, $A = (1, 0)$
(b) $\alpha = 120°$, $A = (-1, 0)$
(c) $\alpha = 30°$, $A = (4, -3)$
(d) $\alpha = 150°$, $A = (-2, -5)$

8. Explain why the four lines ℓ_1: $y = x + 1$, ℓ_2: $y = x - 3$, ℓ_3: $y = 3x + 5$ and ℓ_4: $y = 3x - 5$ enclose a parallelogram. Then find the vertices of this parallelogram.

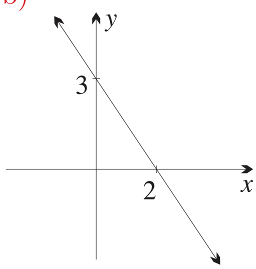
9. Triangle ABC has vertices $A(1, 0)$, $B(6, 5)$ and $C(0, 2)$. Show that it is right-angled, then find the equation of each side.

10. Determine in general form the equation of each straight line sketched:

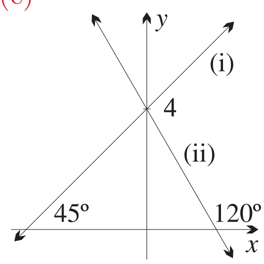
(a)



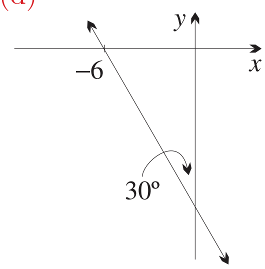
(b)



(c)

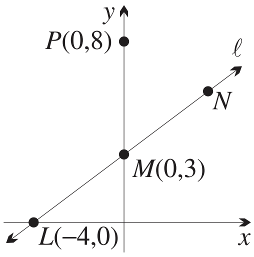


(d)



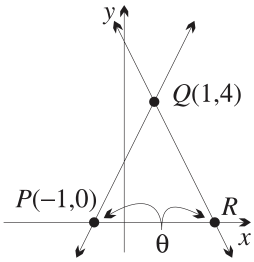
12. (a) On a number plane, plot the points $A(4,3)$, $B(0,-3)$ and $C(4,0)$.
(b) Find the equation of BC. (c) Explain why $OABC$ is a parallelogram.
(d) Find the area of $OABC$ and the length of the diagonal AB.

13. The line ℓ has intercepts at $L(-4,0)$ and $M(0,3)$. N is a point on ℓ and P has coordinates $(0,8)$.



(a) Copy the sketch. (b) Find the equation of ℓ.
(c) Find the lengths of ML and MP and hence show that LMP is an isosceles triangle.
(d) If M is the midpoint of LN, find the coordinates of N.
(e) Show that $\angle NPL = 90°$.
(f) Write down the equation of the circle through N, P and L.

14. The vertices of the triangle are $P(-1,0)$ and $Q(1,4)$ and R, where R lies on the x-axis and $\angle QPR = \angle QRP = \theta$.



(a) Find the coordinates of the midpoint of PQ.
(b) Find the gradient of PQ and show that $\tan\theta = 2$.
(c) Show that PQ has equation $y = 2x + 2$.
(d) Explain why QR has gradient -2, and hence find its equation.
(e) Find the coordinates of R and hence the area of triangle PQR.
(f) Find the length QR, and hence find the perpendicular distance from P to QR.

15. Find k if the lines ℓ_1: $x + 3y + 13 = 0$, ℓ_2: $4x + y - 3 = 0$ and ℓ_3: $kx - y - 10 = 0$ are concurrent. [HINT: Find the point of intersection of ℓ_1 and ℓ_2 and substitute into ℓ_3.]

16. Consider the two lines ℓ_1: $3x + 2y + 4 = 0$ and ℓ_2: $6x + \mu y + \lambda = 0$.
(a) Write down the value of μ if: (i) ℓ_1 is parallel to ℓ_2, (ii) ℓ_1 is perpendicular to ℓ_2.
(b) Given that ℓ_1 and ℓ_2 intersect at a point, what condition must be placed on μ?
(c) Given that ℓ_1 is parallel to ℓ_2, write down the value of λ if: (i) ℓ_1 is the same line as ℓ_2, (ii) the distance between the y-intercepts of the two lines is 2.

17. (a) Write down, in general form, the equation of a line parallel to $2x - 3y + 1 = 0$.
(b) Hence find the equation of the line if it passes through: (i) $(2,2)$ (ii) $(3,-1)$

18. (a) Write down, in general form, the equation of a line perpendicular to $3x + 4y - 3 = 0$.
(b) Hence find the equation of the line if it passes through: (i) $(-1,-4)$ (ii) $(-2,1)$

19. (a) What is the natural domain of the relation $\dfrac{y}{x-1} = 3$? (b) Graph this relation and indicate how it differs from the graph of the straight line $y = 3x - 3$.

20. Explain how $x + y = 1$ may be transformed into $\frac{x}{a} + \frac{y}{b} = 1$ by stretching.

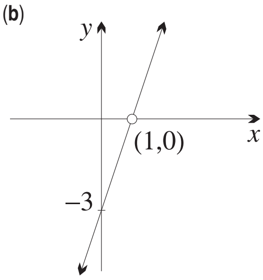
21. Find the point of intersection of $px + qy = 1$ and $qx + py = 1$, and explain why these lines intersect on the line $y = x$.

22. Determine the equation of the line through $M(4,3)$ if M is the midpoint of the intercepts on the x-axis and y-axis. [HINT: This time, let the gradient of the line be m, and begin by writing down the equation of the line in point–gradient form.]

Exercise 5D (Page 173)

1(a) $2x-y-1=0$ **(b)** $x+y-4=0$ **(c)** $3x-y+8=0$
(d) $5x+y=0$ **(e)** $x+3y-8=0$ **(f)** $4x+5y+8=0$
2(a) $y=2x-2$ **(b)** $2x+y-1=0$ **(c)** $x+2y+6=0$
(d) $3y=x+13$
3(a) $\frac{y}{2}-x=1$, $2x-y+2=0$
(b) $\frac{x}{2}+\frac{y}{3}=1$, $3x+2y-6=0$
(c) $-\frac{x}{4}-y=1$, $x+4y+4=0$
(d) $\frac{x}{3}-\frac{y}{3}=1$, $x-y-3=0$
4(c)(i) No, the first two intersect at $(-4,7)$, which does not lie on the third.
(ii) They all meet at $(5,4)$.
5(a) $y=-2x+5$, $y=\frac{1}{2}x+6$
(b) $y=2\frac{1}{2}x-8\frac{1}{2}$, $y=-\frac{2}{5}x+4\frac{1}{5}$
(c) $y=-1\frac{1}{3}x+3$, $y=\frac{3}{4}x+6\frac{1}{2}$
6(a) $3x+2y+1=0$ **(b)** $2x-3y-8=0$
7(a) $x-y-1=0$ **(b)** $\sqrt{3}x+y+\sqrt{3}=0$
(c) $x-y\sqrt{3}-4-3\sqrt{3}=0$
(d) $x+\sqrt{3}y+2+5\sqrt{3}=0$
8 $\ell_1 \parallel \ell_2$, and $\ell_3 \parallel \ell_4$, so there are two pairs of parallel sides. The vertices are
$(-2,-1)$, $(-4,-7)$, $(1,-2)$, $(3,4)$.
9 $m_{BC}\times m_{AC}=-1$ so $BC\perp AC$,
AB: $y=x-1$, BC: $y=\frac{1}{2}x+2$, AC: $y=2-2x$
10(a)(i) $x-3=0$ **(ii)** $y+1=0$
(b) $3x+2y-6=0$ **(c)(i)** $x-y+4=0$
(ii) $\sqrt{3}x+y-4=0$ **(d)** $x\sqrt{3}+y+6\sqrt{3}=0$
11(a) $m_{AC}=\frac{2}{3}$, $\theta \doteqdot 34°$ **(b)** $3y=2x-2$
(c) $D(4,2)$ **(d)** $m_{AC}\times m_{BD}=\frac{2}{3}\times -\frac{3}{2}=-1$, hence they are perpendicular. **(e)** isosceles
(f) area $=\frac{1}{2}\times AC\times BD=\frac{1}{2}\times\sqrt{52}\times\sqrt{52}=26$
(g) $E(8,-4)$

12(b) $3x-4y-12=0$ **(c)** OB and AC are vertical and hence parallel, and from their gradients $(\frac{3}{4})$ OA is parallel to BC.
(d) $12\,\text{units}^2$, $AB=2\sqrt{13}$
13(b) $4y=3x+12$ **(c)** $ML=MP=5$ **(d)** $N(4,6)$
(f) $x^2+(y-3)^2=25$
14(a) $(0,2)$
(d) gradient $=\tan(180°-\theta)=-\tan\theta=-2$ so $2x+y-6=0$ **(e)** $R(3,0)$, hence area $=8\,\text{units}^2$.
(f) $QR=2\sqrt{5}$, $PS=\frac{8}{5}\sqrt{5}$
15 $k=2\frac{1}{2}$
16(a)(i) $\mu=4$ **(ii)** $\mu=-9$
(b) $\mu\neq 4$ **(c)(i)** $\lambda=8$ **(ii)** $\lambda=0$ or 16
17(a) $2x-3y+k=0$
(b)(i) $2x-3y+2=0$ **(ii)** $2x-3y-9=0$
18(a) $4x-3y+k=0$
(b)(i) $4x-3y-8=0$ **(ii)** $4x-3y+11=0$
19(a) $x\neq 1$
(b)



20 Stretch horizontally by a factor of a and vertically by a factor of b.
21 $(\frac{1}{p+q},\frac{1}{p+q})$ The two lines are inverse functions of each other, and so are reflections in the line $y=x$.
22 $3x+4y-24=0$
23 $y-2=m(x+1)$ **(a)** $2x-y=-4$ **(b)** $x-y=-3$
(c) $5x+y=-3$
25(a) $bx+ay=2ab$ **(b)** $bx+2ay=3ab$
(c) $\ell bx+kay=(k+\ell)ab$
28(c)(i) 1 **(ii)** $\frac{1}{13}\sqrt{13}$

Section 4

Solving equations

Learning Goal(s)

Knowledge
Various types of equations and inequations

Skills
Solving linear and quadratic equations, solving quadratic equations

Understanding
Worded problems and converting words into equations and subsequently solving them

By the end of this section am I able to:

1.1 Algebraic techniques

- Solve quadratic equations $ax^2 + bx + c = 0$ by factorisation, completing the square and using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where a, b and c are real numbers and $a \neq 0$

1.2 Linear functions

- Solve linear inequalities and graph the solution on a number line

1.3 Constructing and using functions

- Construct and use linear functions to model and solve problems in real-world situations, identifying the independent and dependent variables and any restrictions on these variables, and justify conclusions in the context of the problem
- Use linear inequalities to model and solve problems in real-world situations, and justify conclusions in the context of the problem
- Solve practical problems involving a pair of simultaneous linear equations both algebraically and graphically, with and without graphing applications, and justify conclusions in the context of the problem
- Construct and use simultaneous equations to model and solve a problem where cost and revenue are represented by linear equations, identify and analyse the break-even point, and justify conclusions in the context of the problem

4.1 Linear equations and inequalities

Laws/Results

Two standard forms of linear equations:

1. Gradient - intercept form

$$y = mx + c$$

2. General form

$$ax + by + c = 0$$

where a is a positive whole number, and b and c are whole numbers.

Vertical/horizontal lines

- Vertical line: $x = a$
- Horizontal line: $y = b$

Further exercises

Ⓐ **Ex 1E** (Pender et al., 2019a)

- Q1-3, 6-10

ⓍⒶ **Ex 1E** (Pender et al., 2019b)

- Q2-4, 7, 8, 11

Ex 5A (Pender et al., 2019b)

- Q3

4.2 Quadratic equations

4.2.1 Completing the square

Example 18

Complete the square for $x^2 + 4x + 1$ and rewrite as $(x - h)^2 + k$.

(See next page for the steps box)

Steps

1. Halve the coefficient of x, and then square it.

2. Complete the square:

Additional questions

1. Complete the square:

(a) x^2+6x+1 (b) x^2-2x-5 (c) a^2+4a (d) $x^2-8x+20$ (e) x^2+x+1 (f) x^2-3x-3 (g) a^2-a (h) $4x^2+4x-5$ (i) $9x^2-24x+14$ (j) $x^2+\frac{1}{2}x+1$ (k) $x^2+\frac{2}{3}x-2$ (l) $x^2-\frac{3}{4}x$

2. Factorise by completing the square:

(a) x^2+2x-8 (b) a^2+2a-1 (c) m^2-4m+1 (d) $4m^2+12m+3$

3. Complete the square, then factorise:

(a) $x^2+2bx+c$ (b) $x^2+\dfrac{b}{a}x+\dfrac{c}{a}$

Answers

1. (a) $(x+3)^2-8$ (b) $(x-1)^2-6$ (c) $(a+2)^2-4$ (d) $(x-4)^2+4$ (e) $\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$ (f) $\left(x-\frac{3}{2}\right)^2-\frac{21}{4}$ (g) $\left(a-\frac{1}{2}\right)^2-\frac{1}{4}$ (h) $(2x+1)^2-6$ (i) $(3x-4)^2-2$ (j) $\left(x+\frac{1}{4}\right)^2+\frac{15}{16}$ (k) $\left(x+\frac{1}{3}\right)^2-\frac{19}{9}$ (l) $\left(x-\frac{3}{8}\right)^2-\frac{9}{64}$ **2.** (a) $(x+4)(x-2)$ (b) $\left(a+1+\sqrt{2}\right)\left(a+1-\sqrt{2}\right)$ (c) $\left(m-2+\sqrt{3}\right)\left(m-2-\sqrt{3}\right)$ (d) $\left(2m+3-\sqrt{6}\right)\left(2m+3+\sqrt{6}\right)$ **3.** (a) $\left(x+b+\sqrt{b^2-c}\right)\left(x+b-\sqrt{b^2-c}\right)$ (b) $\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right)$

Further exercises

(A) **Ex 1H** Pender et al. (2019a)
- Q1-6

(X1) **Ex 1H** Pender et al. (2019b)
- Q1-6

4.2.2 Methods of solution

Laws/Results

Three methods of solution:

1. ....Factorisation............

2. ....Completing............ the ......square...... (rarely used as it is generalised into the method below)

3. ....Quadratic............ ......formula......

Gentle reminder

Difference between an *equation* and *expression*

- An *equation* has an ......equals...... sign in the question. Commences with ......solve...... .
- An *expression* does not have an ......equals...... sign in the question. Usually commences with ......simplify...... , ...expand...... or ......factorise...... .

Important note

⚠ When solving quadratic equations, always solve with ......zero...... as the subject!

Example 19

(a) Fully factorise: $5x^2 + 34x - 7$

(b) Hence solve: $5x^2 + 34x - 7 = 0$

Answer: $\frac{1}{5}$ or -7

Example 20

Solve: $\dfrac{2}{a+3} + \dfrac{a+3}{2} = \dfrac{10}{3}$

Answer: $-\frac{7}{3}$ or 3

Further exercises

Ⓐ **Ex 1F** (Pender et al., 2019a)

- Q2-11

(X1) **Ex 1F** (Pender et al., 2019b)

- Q3-13

4.3 Simultaneous equations

Important note

Three methods of solution:

1. ....Graphical........

- The number of solutions to a pair of equations is geometrically, the number of ....points.... ..of.. ....intersection.... of two corresponding graphs.

2. ....Substitution........

3. ....Elimination........

Example 21

Solve: $\begin{cases} 3x - 2y = 29 \\ 4x + y = 24 \end{cases}$

Answer: $x = 7$, $y = -4$

4.3.1 Break-even analysis

One application of simultaneous equations is to find the break-even point in manufacturing and sales.

Fill in the spaces

- ......Cost...... ......function...... : A formula for the cost to produce a given number of items
- ......Revenue...... ......function...... : the sale price multiplied by the number of units sold

Definition 7

......Break...... - ......even...... ......point...... is when the cost function and revenue function ......intersect...... .

Example 22

(Fitzpatrick & Aus, 2018, p.125) Georgia decides to turn her jewellery-making hobby into a small business. She spends \$120 on equipment and estimates that it costs her \$8 to manufacture each item. She plans to sell the items for \$20 each.

(a) If C is her total cost and R her total revenue, in dollars, set up the cost and revenue functions for the sale of x items.

(b) Determine her break-even point.

(c) Determine her profit if she sells:

(i) 6 items

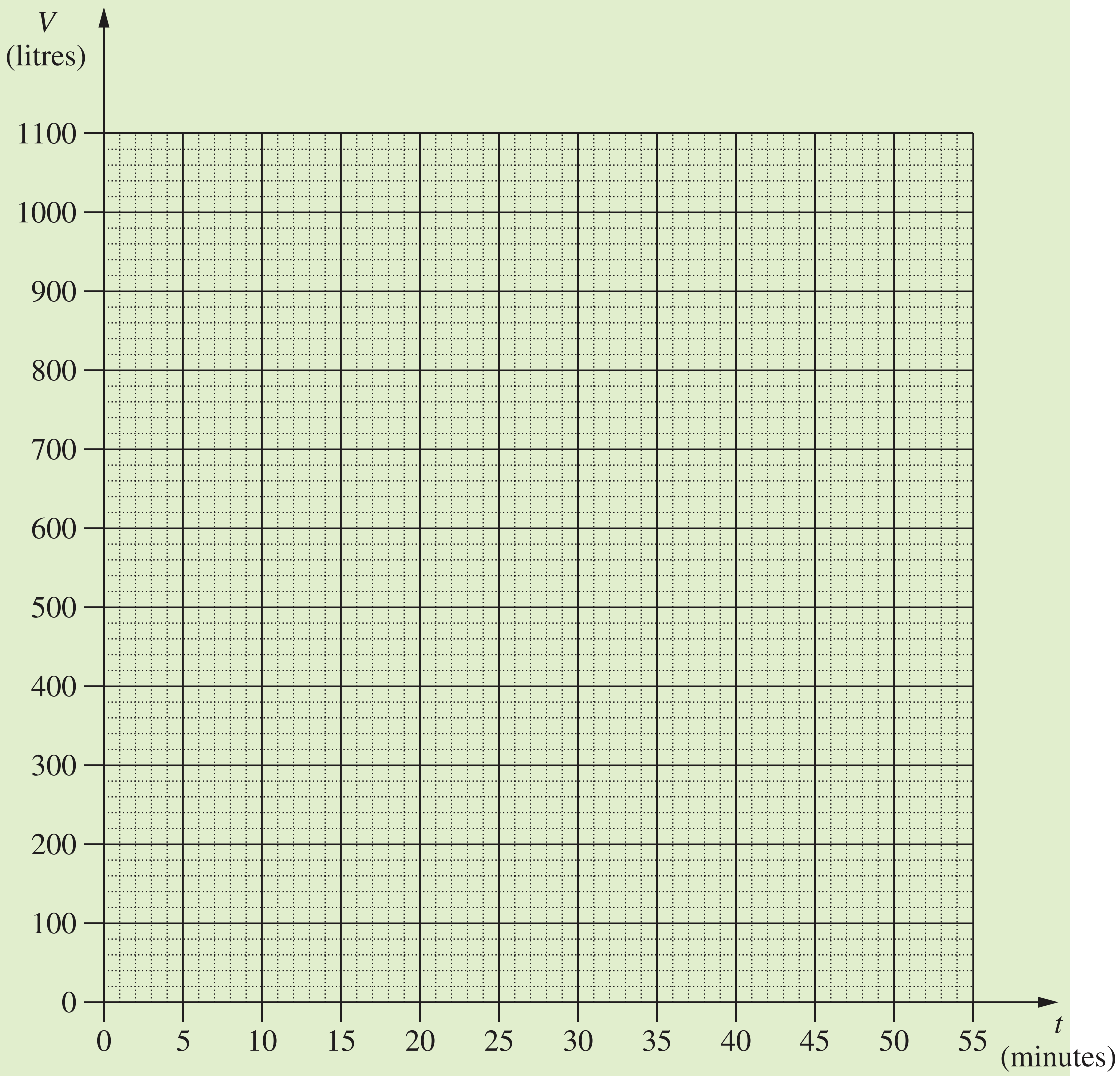
(ii) 50 items

Example 23

[2020 Adv HSC Q11] There are two tanks on a property, Tank A and Tank B. Initially, Tank A holds 1 000 litres of water and Tank B is empty.

(a) Tank A begins to lose water at a constant rate of 20 litres per minute. The volume of water in Tank A is modelled by $V = 1000 - 20t$ where V is the volume in litres and t is the time in minutes from when the tank begins to lose water. **1**

On the grid below, draw the graph of this model and label it as Tank A.



Example 11

[2020 Adv HSC Q11] (*continued from previous page*)

(b) Tank B remains empty until $t = 15$ when water is added to it at a constant rate of 30 litres per minute. **2**

By drawing a line on the grid on the previous page, or otherwise, find the value of t when the two tanks contain the same volume of water.

(c) Using the graphs drawn, or otherwise, find the value of t (where $t > 0$) when the total volume of water in the two tanks is 1000 litres. **1**

Additional questions

Source: Fitzpatrick and Aus (2018, Ex 5.5), unless otherwise stated.

1. For each set of cost and revenue functions (a)-(d), find:

i. the break-even point

ii. the revenue at the break-even point

iii. the profit function, P

(a) $C = 5x + 200, R = 15x$

(b) $C = 0.5x + 100, R = 1.5x$

(c) $C = 15x + 3000, R = 45x$

(d) $C = 0.3x + 5000, R = 1.1x$

2. Julian buys a coffee cart for \$15 000. His repayments work out to be \$80 per day for the first year. He calculates that it will cost him \$2 per cup of coffee for the ingredients. He sells coffee at \$4 per cup and he sells x cups of coffee per day.

(a) Write the cost function C and revenue function R.

(b) What is his break-even point?

(c) Write the profit function P.

(d) What is his profit if he sells 100 cups per day?

3. Maya runs a market stall at the weekends, selling paintings. It costs here \$90 per day for the site. It costs her on average \$4 per painting that she sells and she sells them for an average price of \$10. If she sells x articles each day, find:

(a) the cost function C and revenue function R.

(b) her break-even point

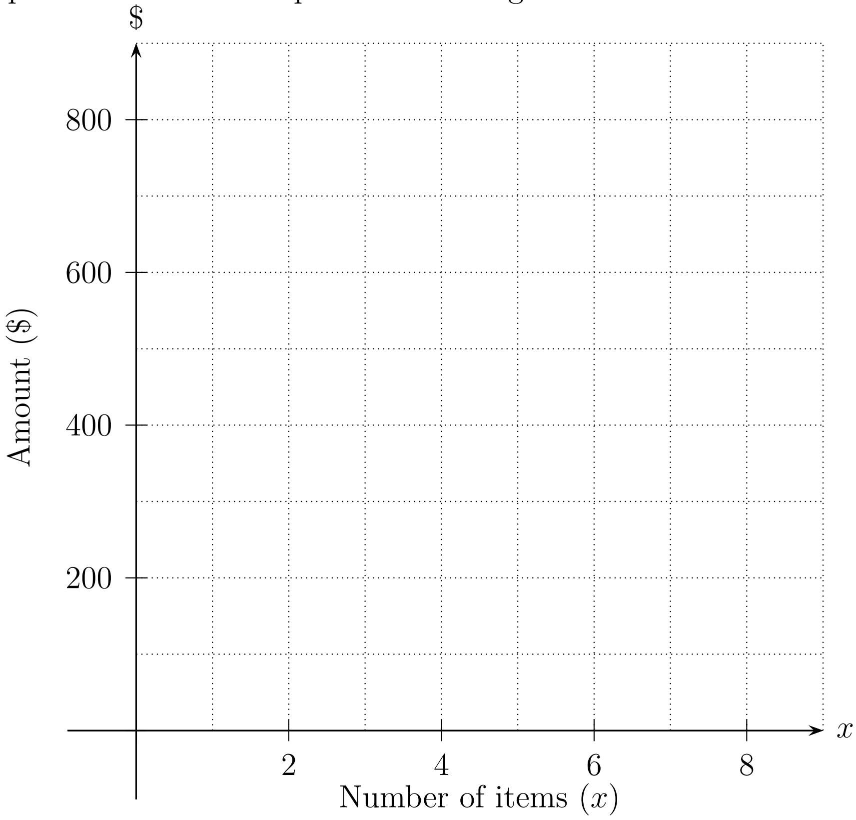
(c) the profit she will make if she sells 40 paintings in a day.

(d) One weekend, the weather is fine on the Saturday and rainy on the Sunday. On Saturday Maya sells 30 paintings, but on Sunday she only sells 10 paintings. What profit (or loss) does she make for the weekend?

4. **[2020 CSSA Adv Trial Q14]** Michael has a small manufacturing business.

The cost of manufacturing is given by the equation $C = 50x + 200$ and the income earned is given by the equation $I = 100x$, where x is the number of items that the business has manufactured.

(a) Graph each of the two equations on the grid below. **2**



(b) How many items need to be manufactured for the business to break even? **1**

Answers

1. (a) i. $x = 20$ ii. $R = \$300$ iii. $P = 10x - 200$ (b) i. $x = 100$ ii. $R = \$150$ iii. $P = x - 100$ (c) i. $x = 100$ ii. $R = \$4500$ iii. $P = 30x - 3000$ (d) i. $x = 6250$ ii. $R = \$6875$ iii. $P = 0.8x - 5000$ **2.** (a) $C = 2x + 80, R = 4x$ (b) $x = 40$ (c) $P = 2x - 80$ **3.** (a) $C = 4x + 90, R = 10x$ (b) $x = 15$ (c) $P = 6x - 90, x = 40, P = \$150$ (d) Saturday $x = 30, P = \$90$/ Sunday $x = 10, P = -\$30$/ Profit for the weekend: \$60 **4.** 4 items

Further exercises

Ⓐ **Ex 1G** (Pender et al., 2019a)
- Q3-5

Ⓧ1 **Ex 1G** (Pender et al., 2019b)
- Q3-5

Example 12

[**Ex 8B Q20**] Solve simultaneously:

$$\begin{cases} 7^{2x-y} = 49 \\ 2^{x+y} = 128 \end{cases}$$

4.3.2 Quadratic equations arising from simultaneous equations

Example 13

Solve: $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$

Answer: $x = 2$, $y = 4$ or $x = -1$, $y = 1$

Important note

After substituting, what type of equation shows up?

Example 14

Solve: $\begin{cases} x^2 + y^2 = 53 \\ x^2 - y^2 = 45 \end{cases}$

Answer: $x = 7$, $y = \pm 2$ or $x = -7$, $y \pm 2$

Section 5

Discriminant of a quadratic equation

Learning Goal(s)

Knowledge
Various types of equations and inequations

Skills
Solving linear and quadratic equations, solving quadratic equations

Understanding
Worded problems and converting words into equations and subsequently solving them

By the end of this section am I able to:

1.1 Algebraic techniques

- Define the discriminant as $\Delta = b^2 - 4ac$ and use it to solve problems involving the number of real roots of a quadratic equation and determine the conditions for roots to be equal, distinct, real or rational

Fill in the spaces

1. What is the relationship and difference between (5.1) and (5.2)?

$$y = 3x^2 - 5x + 5 \tag{5.1}$$

$$3x^2 - 5x + 5 = 0 \tag{5.2}$$

- (5.1) is a *quadratic* ....function........ , where a ....parabola........ can then be drawn
- (5.2) is a *quadratic* ....equation........ .
  - ....Solving........ (5.2) may result in finding the ....roots........ to the equation, and ..x.. ....intercept(s)........ of the quadratic ....function........ .

2. What needs to be done in order to obtain (5.2) from (5.1)?

- Set ....$y = 0$.... in (5.1).

5.1 Uniqueness of roots

Definition 8

The **discriminant** (Δ) of a quadratic equation $ax^2 + bx + c = 0$:

$$\Delta = \underline{\quad b^2 - 4ac \quad}$$

Laws/Results

Given the roots of a quadratic equation are located at: $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

- $\Delta = 0$: ...one... distinct (unique), ...real... root. (Double root)

  Pictorial example

- $\Delta > 0$: ...two... distinct (unique), ...real... roots.

  ➜ Rational roots when ...$\sqrt{\Delta}$... is rational, i.e. Δ is a ...perfect... ...square... .

  Pictorial example

- $\Delta < 0$: ...no... ...real... roots. (...two... ...non... ...real... roots)

  Pictorial example

Example 15

Use the discriminant to determine the number of solutions to x to the following:

(a) $4x^2 - 4x + 1 = 0.$

(b) $3x^2 - 5x + 5 = 0$

Example 16

For what values of λ does $x^2 - (\lambda + 5)x + 9 = 0$ have:

Answer: (a) $1, -11$ (b) $\lambda \in (-11, 1)$

(a) equal roots

(b) no real roots?

Example 17

Find the values of k for which $y = x^2 + (k-1)x - (2k+1)$ has unreal roots.

Answer: $k \in (-5, -1)$

Example 18

Show that the roots of the equation

$$4(m+1)x^2 - 4(m-1)x - 3 = 0 \qquad (m \neq -1)$$

are always real, for all $m \in \mathbb{R}$.

Example 19

[Ex 8F Q22/1988 HSC] Find, as a relation between k, ℓ and m, the condition for the quadratic equation in x

$$\left(k^2 + \ell^2\right) x^2 + 2\ell(k+m)x + \left(\ell^2 + m^2\right) = 0$$

to have real roots. Simplify your answer as far as possible.

Example 20

[2006 St George Girls' 2U]

(i) State the condition necessary for a quadratic equation to have rational roots. **1**

(ii) Prove that the equation $3px^2 = 2px + 3qx - 2q$, where p and q are rational, has rational roots for all values of p and q. **4**

(iii) What can be concluded about the number of roots if $p = \dfrac{3q}{2}$? **1**

Further exercises

Ⓐ **Ex 10F** (Pender et al., 2009)

- Q1-8, 11, 13-15

Ⓧ1 **Ex 8F** (Pender et al., 1999)

- Q2-5 LC
- Q8-11, 21-22

Exercise 8F

1. Describe the roots of quadratic equations with rational coefficients that have the following discriminants. If the roots are real, state whether they are equal or unequal, rational or irrational.

(a) $\Delta = 7$
(b) $\Delta = -9$
(c) $\Delta = 0$
(d) $\Delta = 64$
(e) $\Delta = \frac{4}{9}$
(f) $\Delta = -0{\cdot}3$

2. Find the discriminant Δ of each equation. Hence state how many roots there are, and whether or not they are rational.

(a) $x^2 - 4x + 3 = 0$
(b) $2x^2 - 3x + 5 = 0$
(c) $x^2 - 6x + 9 = 0$
(d) $x^2 + 2x - 7 = 0$
(e) $6x^2 + 11x - 10 = 0$
(f) $9x^2 - 1 = 0$

NOTE: In questions 3–7, first find the discriminant Δ, then answer the question. If it is necessary to solve a quadratic inequality, this should be done by sketching a graph.

3. Find g, if the following quadratic functions have exactly one distinct zero:

(a) $y = x^2 + 10x + g$
(b) $y = gx^2 - 4x + 1$
(c) $y = 2x^2 - 3x + (g + 1)$
(d) $y = (g - 2)x^2 + 6x + 1$
(e) $y = gx^2 - gx + 1$
(f) $y = gx^2 + 7x + g$
(g) $y = 4x^2 + 4gx + (6g + 7)$
(h) $y = 9x^2 - 2(g + 1)x + 1$

4. Find the values of k for which the roots of the following equations are real numbers:

(a) $x^2 + 2x + k = 0$
(b) $kx^2 - 8x + 2 = 0$
(c) $3x^2 - 4x + (k + 1) = 0$
(d) $(2k - 1)x^2 - 5x + 2 = 0$
(e) $x^2 + kx + 4 = 0$
(f) $x^2 - 3kx + 9 = 0$
(g) $4x^2 - (6 + k)x + 1 = 0$
(h) $9x^2 + (k - 6)x + 1 = 0$

5. Find the values of ℓ for which the following quadratic functions have no real zeroes:

(a) $y = x^2 + \ell x + 4$
(b) $y = \ell x^2 + 6x + \ell$
(c) $y = x^2 + (\ell + 1)x + 4$
(d) $y = 9x^2 - 4(\ell - 1)x - \ell$
(e) $y = \ell x^2 - 4\ell x - (\ell - 5)$
(f) $y = (\ell - 3)x^2 + 2\ell x + (\ell + 2)$

6. (a) Show that the x-coordinates of the points of intersection of the circle $x^2 + y^2 = 4$ and the line $y = x + 1$ satisfy the equation $2x^2 + 2x - 3 = 0$.

(b) Evaluate the discriminant Δ and explain why this shows that there are two points of intersection.

7. Using the method outlined in the previous question, determine how many times the line and circle intersect in each case:

(a) $x^2 + y^2 = 9$, $y = 2 - x$
(b) $x^2 + y^2 = 1$, $y = x + 2$
(c) $x^2 + y^2 = 5$, $y = -2x + 5$
(d) $(x - 3)^2 + y^2 = 4$, $y = x - 4$

DEVELOPMENT

8. Find Δ for each equation. By writing Δ as a perfect square, show that each equation has rational roots for all rational values of m and n:

(a) $4x^2 + (m - 4)x - m = 0$
(b) $(m - 1)x^2 + mx + 1 = 0$
(c) $mx^2 + (2m + n)x + 2n = 0$
(d) $2mx^2 - (4m + 1)x + 2 = 0$
(e) $2(m - 2)x^2 + (6 - 7m)x + 6m = 0$
(f) $(4m + 1)x^2 - 2(m + 1)x + (1 - 2m) = 0$

9. Prove that the roots of the following equations are real and distinct for all real values of λ. [HINT: Find Δ and write it in such a way that it is obviously positive.]

(a) $x^2 + \lambda x - 1 = 0$

(b) $3x^2 + 2\lambda x - 4 = 0$

(c) $\lambda x^2 - (\lambda + 4)x + 2 = 0$

(d) $x^2 + (\lambda + 1)x + (\lambda - 2) = 0$

NOTE: In the following questions you may need to rearrange the quadratic equation into the form $ax^2 + bx + c = 0$ before finding Δ.

10. Find the values of m for which the roots of the quadratic equation:

(a) $1 - 3x - mx^2 = 0$ are real and distinct,

(b) $2x^2 + 4x + 5 = 3x + m$ are real and equal,

(c) $x(x - 2m) = m - 2x - 3$ are unreal,

(d) $12m(x^2 - 2x) + 12(2x^2 + x) = 38m + 11$ are real.

11. Show that the roots of $(x - a)(x - b) = c^2$ are always real, where a, b and c are real.

12. (a) For what values of b is the line $y = x + b$ a tangent to the curve $y = 2x^2 - 7x + 4$?

(b) The line $2x + y + b = 0$ is a tangent to $y = 2x^2 + 3x + 1$. Find the value of b.

(c) The line $y = mx + 4$ is a tangent to $y = 3x^2 + 5x + 7$. Find the value of m.

13. Find the equation of the tangent to the parabola $y = x^2 - 5x - 3$ that is parallel to the line $3x - y - 7 = 0$.

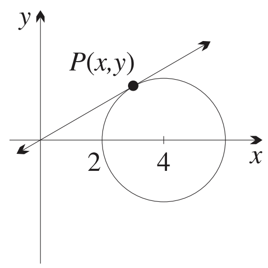
14. Find the gradients of the lines that pass through the point $(1, 7)$ and are tangent to the parabola $y = (2 - x)(1 + 3x)$.

15. The line $y = 4x - 7$ is tangent to a parabola that has a y-intercept of -3 and the line $x = \frac{1}{2}$ as its axis of symmetry. Find the equation of the parabola.

16. How many horizontal tangents may be drawn to each of the following cubic functions? [HINT: You will need to differentiate and set the derivative equal to zero, then use the discriminant to find how many solutions this equation has.]

(a) $y = x^3 + 5x^2 - 8x + 7$ (b) $y = 3x^3 - 3x^2 + x - 1$ (c) $y = \frac{1}{3}x^3 + x^2 + 5x + 11$

17. If in $y = ax^2 + bx + c$ we find $ac < 0$, explain why the graph of the parabola must have two x-intercepts.



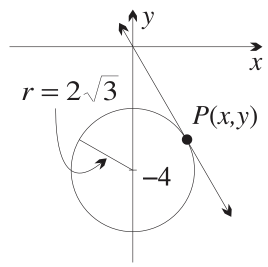
18. (a) Write down the equation of the circle in the diagram.

(b) Write down the equation of the line through the origin with gradient m.

(c) By solving the circle and the line simultaneously, show that the x-coordinate of the point P in the diagram satisfies the equation $(m^2 + 1)x^2 - 8x + 12 = 0$.

(d) Use the theory of the discriminant to find the value of m.

(e) Hence or otherwise find the coordinates of P.



19. Use the method outlined in the previous question to find the gradient of the line in the diagram and hence the coordinates of the point P.

20. Use the discriminant to find the gradients of the lines that pass through the point $(7, 1)$ and are tangent to the circle $x^2 + y^2 = 25$.

Exercise **8F** (Page 302)

1(a) irrational, unequal **(b)** unreal (that is, no roots) **(c)** rational, equal (that is, a single rational root) **(d)** rational, unequal **(e)** rational, unequal **(f)** unreal

2(a) $\Delta = 4$, two rational roots
(b) $\Delta = -31$, no roots
(c) $\Delta = 0$, one rational root
(d) $\Delta = 32$, two irrational roots
(e) $\Delta = 361 = 19^2$, two rational roots
(f) $\Delta = 36$, two rational roots

3(a) $\Delta = 100 - 4g$, $g = 25$ **(b)** $\Delta = 16 - 4g$, $g = 4$
(c) $\Delta = 1 - 8g$, $g = \frac{1}{8}$ **(d)** $\Delta = 44 - 4g$, $g = 11$
(e) $\Delta = g^2 - 4g$, $g = 4$ (If $g = 0$, then $y = 1$ for all x, and so y is never zero.)
(f) $\Delta = 49 - 4g^2$, $g = \frac{7}{2}$ or $-\frac{7}{2}$
(g) $\Delta = 16(g^2 - 6g - 7)$, $g = -1$ or 7
(h) $\Delta = 4(g^2 + 2g - 8)$, $g = -4$ or 2

4(a) $\Delta = 4 - 4k$, $k \le 1$ **(b)** $\Delta = 64 - 8k$, $k \le 8$
(c) $\Delta = 4 - 12k$, $k \le \frac{1}{3}$ **(d)** $\Delta = 33 - 16k$, $k \le \frac{33}{16}$
(e) $\Delta = k^2 - 16$, $k \le -4$ or $k \ge 4$
(f) $\Delta = 9k^2 - 36$, $k \le -2$ or $k \ge 2$
(g) $\Delta = k^2 + 12k + 20$, $k \le -10$ or $k \ge -2$
(h) $\Delta = k^2 - 12k$, $k \le 0$ or $k \ge 12$

5(a) $-4 < \ell < 4$ **(b)** $\ell < -3$ or $\ell > 3$
(c) $-5 < \ell < 3$ **(d)** no values **(e)** $0 < \ell < 1$
(f) $\ell < -6$

6(b) $\Delta = 28$. Since $\Delta > 0$, the quadratic equation has two roots.

7(a) They intersect twice.
(b) They do not intersect.
(c) They intersect once.
(d) They intersect twice.

8(a) $\Delta = (m + 4)^2$ **(b)** $\Delta = (m - 2)^2$
(c) $\Delta = (2m - n)^2$ **(d)** $\Delta = (4m - 1)^2$
(e) $\Delta = (m - 6)^2$ **(f)** $\Delta = 36m^2$

9(a) $\Delta = \lambda^2 + 4$ **(b)** $\Delta = 4\lambda^2 + 48$ **(c)** $\Delta = \lambda^2 + 16$
(d) $\Delta = (\lambda - 1)^2 + 8$

10(a) $m > -\frac{9}{4}$ **(b)** $m = \frac{39}{8}$ **(c)** $-1 < m < 2$
(d) $m \le -1$ or $m \ge -\frac{1}{2}$

12(a) -4 **(b)** $\frac{17}{8}$ **(c)** -1 or 11

13 $3x - y - 19 = 0$

14 -7 and 5

15 $y = 4x^2 - 4x - 3$

16(a) 2 **(b)** 1 **(c)** 0

17 If $ac < 0$, then $\Delta > 0$.

18(a) $(x - 4)^2 + y^2 = 4$ **(b)** $y = mx$
(d) $m = \dfrac{1}{\sqrt{3}}$ or $m = -\dfrac{1}{\sqrt{3}}$
(e) $(3, \sqrt{3})$ or $(3, -\sqrt{3})$

19 $m = \frac{1}{\sqrt{3}}$ or $m = -\frac{1}{\sqrt{3}}$, $P(\sqrt{3}, -1)$ or $P(-\sqrt{3}, -1)$

20 $\frac{4}{3}$ and $-\frac{3}{4}$

21(a) $-\frac{4}{3} < a < 1$ **(b)** $b \ne -5$ **(c)** $g = 3$ or $-\frac{13}{29}$
(d) $-1 < k < 14$

22(b) $b^2 = ac$

5.2 Intersections of graphs and other geometric results

Laws/Results

Geometric results when a quadratic arises from solving a pair of simultaneous equations:

- $\Delta = 0$: curves will ....touch.... .
  - If one of the curves is a line, the line will be a ....tangent.... at the point of contact.

  Example

- $\Delta < 0$: curves will ....not.... ....intersect.... .

  Example

- $\Delta > 0$: curves will ....intersect.... ....twice.... .

  Example

Example 21

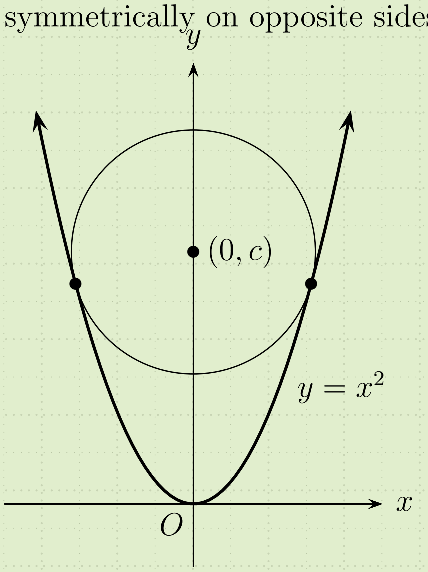
Use the discriminant to find the equations of the lines through $A(3,-3)$ which are tangent to the rectangular hyperbola $y=\dfrac{3}{x}$.

Example 22

For what values of m does the line $y=mx$ not meet the rectangular hyperbola $y=\dfrac{3}{x-2}$?

Example 23

[2012 HSC Q16] The circle $x^2 + (y - c)^2 = r^2$ where $c > 0$ and $r > 0$, lies inside the parabola $y = x^2$. The circle touches the parabola at exactly two points located symmetrically on opposite sides of the y axis, as shown in the diagram.



i. Show that $4c = 1 + 4r^2$. **2**

ii. Deduce that $c > \dfrac{1}{2}$. **1**

Further exercises

Ⓐ **Ex 10F** (Pender et al., 2009)

- Q9-10, 12

ⓍⒾ **Ex 8F** (Pender et al., 1999)

- Q6-7
- Q12-20

5.3 Definite/indefinite quadratics

Definition 9

Given a quadratic equation $ax^2 + bx + c = 0$ and $\Delta = b^2 - 4ac$,

- $\Delta < 0$ and $a > 0$: ....positive........ ....definite........ .

  Example

- $\Delta < 0$ and $a < 0$: ....positive........ ....definite........ .

  Example

All other variations are *indefinite*.

Further exercises

Ⓐ **Ex 10G** (Pender et al., 2009)

- All questions

ⓍⅠ **Ex 8G** (Pender et al., 1999)

- Q2-5 LC
- Q8, 12-14
- ⏯ Q16

Exercise 8G

1. Use a graph to solve the following quadratic inequations:

(a) $x^2 \geq x$
(b) $7 - x^2 > 0$
(c) $x^2 + 9 > 6x$
(d) $(x+1)^2 \leq 34$
(e) $3x^2 + 5x - 2 \leq 0$
(f) $-2x^2 + 13x \geq 15$
(g) $-5 > 4x(2-x)$
(h) $x^2 + 4x + 5 \leq 0$
(i) $-2x^2 - 3x - 3 < 0$

2. Evaluate the discriminant and look carefully at the coefficient of x^2 to determine whether the following functions are positive definite, negative definite or indefinite:

(a) $y = 2x^2 - 5x + 7$
(b) $y = x^2 - 4x + 4$
(c) $y = 5x - x^2 - 9$
(d) $y = -x^2 + 7x - 3$
(e) $y = 25 - 20x + 4x^2$
(f) $y = 3x + 2x^2 + 11$

3. Find the discriminant as a function of k, and hence find the values of k for which the following expressions are: (i) positive definite, (ii) indefinite.

(a) $2x^2 - 5x + 4k$
(b) $2x^2 - kx + 8$
(c) $3x^2 + (12-k)x + 12$
(d) $x^2 - 2(k-3)x + (k-1)$

4. Find the discriminant as a function of m, and hence find the values of m for which these expressions are: (iii) negative definite, (iv) indefinite.

(a) $-x^2 + mx - 4$
(b) $-2x^2 + 3x - m$
(c) $-x^2 + (m-2)x - 25$
(d) $-4x^2 + 4(m+1)x - (4m+1)$

5. Find the values of ℓ that will make each quadratic below a perfect square:

(a) $x^2 - 2\ell x + 16$
(b) $2\ell x^2 + 2\ell x + 1$
(c) $(5\ell - 1)x^2 - 4x + (2\ell - 1)$
(d) $(4\ell + 1)x^2 - 6\ell x + 4$

DEVELOPMENT

6. Show that the discriminant of $(3k-5)x^2 + 2(4-k)x + 4 = 0$ is $\Delta = 4(k^2 - 20k + 36)$, and hence find the values of k for which $(3k-5)x^2 + 2(4-k)x + 4 = 0$ is:

(a) positive definite,
(b) negative definite,
(c) indefinite.

7. (a) Sketch a graph of the function $y = \dfrac{(x-1)(x+2)}{x(x+4)}$.

(b) Prove that the roots of the equation $kx(x+4) = (x-1)(x+2)$ are always real.

(c) How can you establish this result from the graph you have drawn?

8. Find the domain of each of the following expressions:

(a) $\sqrt{x^2 - 5x + 1}$
(b) $\sqrt{2x - 3 - x^2}$
(c) $\sqrt{2x^2 - 9x + 4}$
(d) $\dfrac{x+2}{\sqrt{2x - x^2}}$
(e) $\sqrt{6 + 5x - 4x^2}$
(f) $\dfrac{(x-1)(2x+3)}{\sqrt{x^2 - 9}}$

9. State in terms of a, b and c the conditions necessary for $ax^2 + 2bx + 3c$ to be:

(a) positive definite,
(b) negative definite,
(c) indefinite.

10. Sketch a possible graph of the quadratic function $y = ax^2 + bx + c$ if:

(a) $a > 0$, $b > 0$, $c > 0$ and $b^2 - 4ac > 0$
(b) $a < 0$, $c < 0$ and $b = 0$
(c) $a > 0$, $b < 0$, $c > 0$ and $b^2 = 4ac$
(d) $a > 0$, $b < 0$ and $b^2 - 4ac < 0$

11. State in terms of b and c the condition for the roots of $x^2 + 2bx + 3c = 0$ to be:

(a) equal,
(b) real and distinct,
(c) unreal,
(d) opposite in sign,
(e) distinct and positive,
(f) distinct and negative.

12. The expression $x^2 - xy - 2y^2 + x + 7y - 5$ can be treated as a function in x with y as an arbitrary constant. Show that it is positive definite when $1 < y < \frac{7}{3}$.

13. Find the range of values of x for which the equation in y, $2x^2 - 3xy + y^2 - 5x + 11 = 0$ will have real roots. Find also for what values of y the equation in x will have real roots.

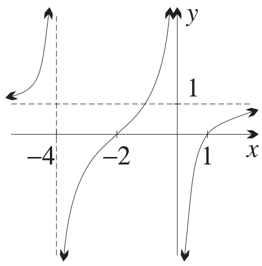
14. The expression $3x^2 + 2xy - 8y^2 - 8x + 14y - 3$ can be treated as a function in x with y as an arbitrary constant or as a function in y with x as an arbitrary constant. Show that in either case the expression is indefinite. Factor the expression.

15. Find the values of λ for which $4a^2 - 10ab + 10b^2 + \lambda(3a^2 - 10ab + 3b^2)$ is a perfect square.

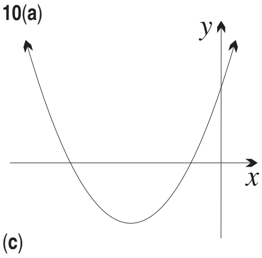
Exercise **8G** (Page 305)

1(a) $x \le 0$ or $x \ge 1$ **(b)** $-\sqrt{7} < x < \sqrt{7}$ **(c)** $x \ne 3$
(d) $-1 - \sqrt{34} \le x \le -1 + \sqrt{34}$ **(e)** $-2 \le x \le \frac{1}{3}$
(f) $\frac{3}{2} \le x \le 5$ **(g)** $x < -\frac{1}{2}$ or $x > \frac{5}{2}$
(h) There are no solutions.
(i) All real numbers are solutions.
2(a) positive definite **(b)** indefinite
(c) negative definite **(d)** indefinite
(e) indefinite **(f)** positive definite
3(a)(i) $k > \frac{25}{32}$ **(ii)** $k \le \frac{25}{32}$
(b)(i) $-8 < k < 8$ **(ii)** $k \le -8$ or $k \ge 8$
(c)(i) $0 < k < 24$ **(ii)** $k \le 0$ or $k \ge 24$
(d)(i) $2 < k < 5$ **(ii)** $k \le 2$ or $k \ge 5$
4(a)(i) $-4 < m < 4$ **(ii)** $m \le -4$ or $m \ge 4$
(b)(i) $m > \frac{9}{8}$ **(ii)** $m \le \frac{9}{8}$
(c)(i) $-8 < m < 12$ **(ii)** $m \le -8$ or $m \ge 12$
(d)(i) $0 < m < 2$ **(ii)** $m \le 0$ or $m \ge 2$
5(a) -4 and 4
(b) 2. (When $\ell = 0$, it is not a quadratic.)
(c) 1. (When $\ell = -\frac{3}{10}$, the expression becomes $-\frac{1}{10}(5x - 4)^2$, which is a multiple of a perfect square, but is not itself a perfect square.)
(d) $-\frac{2}{9}$ and 2
6(a) $2 < k < 18$ **(b)** no values
(c) $k \le 2$ or $k \ge 18$, but $k \ne \frac{5}{3}$

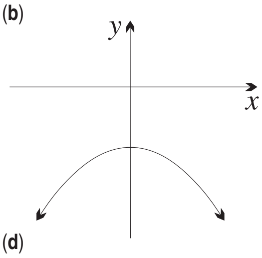
7(a)



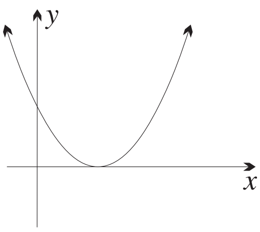
(c) Whatever the value of k, every horizontal line $y = k$ intersects the graph.
8(a) $x \le \frac{1}{2}(5 - \sqrt{21}\,)$ or $x \ge \frac{1}{2}(5 + \sqrt{21}\,)$
(b) no values **(c)** $x \le \frac{1}{2}$ or $x \ge 4$
(d) $0 < x < 2$ **(e)** $-\frac{3}{4} \le x \le 2$
(f) $x < -3$ or $x > 3$
9(a) $a > 0$ and $b^2 < 3ac$
(b) $a < 0$ and $b^2 < 3ac$ **(c)** $b^2 \ge 3ac$
10(a)



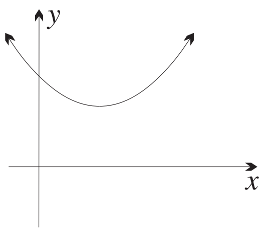
(b)



(c)



(d)



11(a) $b^2 = 3c$ **(b)** $b^2 > 3c$ **(c)** $b^2 < 3c$ **(d)** $c < 0$
(e) $c > 0$ and $b < 0$ (and $b^2 > 3c$)
(f) $c > 0$ and $b > 0$ (and $b^2 > 3c$)
13 $x \le -22$ or $x \ge 2$, $y \le -15 - 12\sqrt{2}$ or $y \ge -15 + 12\sqrt{2}$
14 $(3x - 4y + 1)(x + 2y - 3)$
15 $-\frac{5}{4}$ and $\frac{3}{4}$
16 9. Note that a^2 cannot be negative.
17(b) $(x - 1)^2 + (x + 3)^2$

NESA Reference Sheet – Mathematics Advanced

Mathematics Advanced

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}\left(a + b\right)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Financial mathematics

$$FV = PV\left(1 + r\right)^n$$

Sequences and series

$$a_n = a + \left(n - 1\right)d$$

$$S_n = \frac{n}{2}\left[2a + \left(n - 1\right)d\right] = \frac{n}{2}\left(a + a_n\right)$$

$$a_n = ar^{n-1}$$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r},\ r \neq 1$$

$$S_\infty = \frac{a}{1 - r},\ \left|r\right| < 1$$

Logarithmic and exponential functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Relations

$$\left(x - a\right)^2 + \left(y - b\right)^2 = r^2$$

Trigonometric functions

$$\sin A = \frac{\text{opp}}{\text{hyp}},\quad \cos A = \frac{\text{adj}}{\text{hyp}},\quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

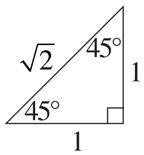
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

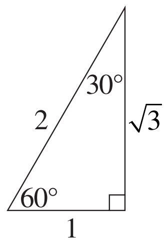
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A},\ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A},\ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A},\ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Differential calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^x$	$\dfrac{dy}{dx} = (\ln a)a^x$
$y = \log_a x$	$\dfrac{dy}{dx} = \dfrac{1}{x\ln a}$

Integral calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int_a^b f(x)\,dx$$

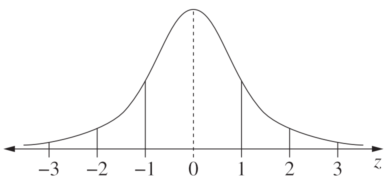
$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Statistical analysis

$$z = \frac{x - \mu}{\sigma}$$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

Discrete random variables

$$\mathrm{E}(X) = \mu$$

$$\mathrm{Var}(X) = \mathrm{E}\left[(X - \mu)^2\right]$$

Continuous random variables

$$P(X \le x) = F(x) = \int_a^x f(t)\,dt$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

$$\mathrm{E}(X) = \mu = \int_a^b xf(x)\,dx$$

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \mu^2 = \int_a^b x^2 f(x)\,dx - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0$$

References

Fitzpatrick, J. B., & Aus, B. (2018). *New Senior Mathematics – Advanced Course for Years 11 & 12* (3rd ed.). Pearson Education.

Grove, M. (2010). *Maths in focus: mathematics extension preliminary course* (E. Bron, Ed.). McGraw-Hill Australia Pty Ltd.

Lowe, R., & Lam, H. (2010). *A complete course on factorisation.* Normanhurst Boys High School Edmodo site. (Compiled by R. Lowe. Typeset in 2010 and edited in 2013 by H. Lam)

Pender, W., Sadler, D., Shea, J., & Ward, D. (1999). *Cambridge Mathematics 3 Unit Year 11* (1st ed.). Cambridge University Press.

Pender, W., Sadler, D., Shea, J., & Ward, D. (2009). *Cambridge Mathematics 2 Unit Year 11* (2nd ed.). Cambridge University Press.

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019a). *CambridgeMATHS Stage 6 Mathematics Advanced Year 11* (1st ed.). Cambridge Education.

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019b). *CambridgeMATHS Stage 6 Mathematics Extension 1 Year 11* (1st ed.). Cambridge Education.