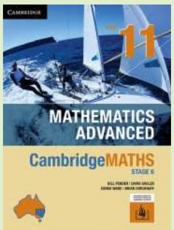
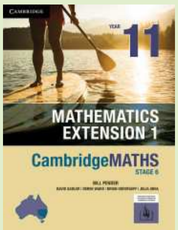
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS ADVANCED
(INCORPORATING EXTENSION 1)
YEAR 11 COURSE

Topic summary and exercises:

With references to





(A) **Functions & Graphs**

Name: ....

Initial version by H. Lam, 2018. Last updated January 16, 2026.
Based on the work from the legacy syllabuses by R. Trenwith, 1995–2010, subsequently maintained by H. Lam, 2011-18.
Additional editing by I. Ham and M. Ho, 2019-2020, A. Sun in late 2020, and H. Lam again for 2026 new syllabuses
Various corrections by students & members of the Mathematics Departments at North Sydney Boys and Normanhurst Boys High Schools.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at http://www.flaticon.com, used under CC BY 2.0.

Symbols used

Beware! Heed warning.

(A) Mathematics Advanced content, for `MAA` coded class

(X1) Mathematics Extension 1 content, for `MAX` coded class

Literacy: note new word/phrase.

Facts/formulae to memorise.

On the course Reference Sheet.

ICT usage

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Content outcomes addressed

MAV-11-02 Uses functions and relations to model, analyse and solve problems

MAV-11-03 Analyses and solves algebraic and graphical problems involving transformations of functions and relations

MAV-11-08 Analyses graphs of exponential and logarithmic functions

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Further exercises from
  - (A) *Cambridge Year 11 2 Unit* (Pender, Sadler, Shea, & Ward, 2009)
  - (A) *Cambridge Year 11 Advanced* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019a)
  - (X1) *Cambridge Year 11 3 Unit* (Pender, Sadler, Shea, & Ward, 1999)
  - (X1) *Cambridge Year 11 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019b)

  will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Learning intentions & outcomes

✓ **Content**

☐ 2.1 Introduction to functions and relations

- Describe a relation between two sets as an association between the elements of one set and the elements of the other set
- Define a real function f of a real variable x as a relation where each element x of a given set is associated with exactly one element y of the second set
- Recognise that a relation is a rule which can be represented by an algebraic formula, a table of values, a set of ordered pairs (x, y) or a graph
- Use the notation $f(x)$ to identify the unique value of y associated with x when working with functions, and refer to it as the value of f at x
- Refer to x in a function $y = f(x)$ as the independent variable of the function, and refer to y as the dependent variable
- Substitute numeric and algebraic expressions into the formulas of functions
- Apply the vertical line test on the graph of a relation to determine whether it represents a function
- Define the domain of a function f as the set of real numbers on which f is defined
- Define the range of a function f as the set of values of $f(x)$ obtained as x varies over the domain of f
- Refer to a value in the domain of a function at which the function is 0 as a zero of the function
- Recognise that the x-intercepts of the graph of $y = f(x)$ are its zeroes, the solutions of $f(x) = 0$, and that the y-intercept is $f(0)$

✓ Content

☐ 2.2 Properties of functions, relations and graphs

- Extend the definitions of domain and range to relations
- Recognise domains and ranges of functions and relations given in interval notation, as inequalities and as worded descriptions
- Determine and describe the domain and range of functions and relations, using interval notation, inequalities or worded descriptions
- Define a function to be even if its graph is unchanged under reflection in the y-axis, and odd if its graph is unchanged under rotation of 180° about the origin
- Develop and use the tests that a function $f(x)$ is odd if $f(-x) = -f(x)$ and a function $f(x)$ is even if $f(-x) = f(x)$
- Solve problems involving even and odd functions
- Use the composite function $f\Big(g(x)\Big)$, where the output of $g(x)$ becomes the input of $f(x)$
- Determine the equations of composite functions

✓ **Content**

2.3 Quadratic and cubic functions

- Identify the x-intercepts of a parabola whose quadratic function is expressed in factored form
- Use the discriminant to determine the number of x-intercepts on a parabola and justify its position in relation to the x-axis
- Show by completing the square on the general quadratic $y = ax^2 + bx + c$ that the axis of symmetry is $x = -\frac{b}{2a}$ where a, b and c are constants
- Identify the axis of symmetry and vertex of a parabola by completing the square on its quadratic function
- Choose and apply appropriate techniques to graph a parabola of the form $y = ax^2 + bx + c$ by identifying its x-intercepts if they exist, its y-intercept, its axis of symmetry using $x = -\frac{b}{2a}$ and its vertex
- Find the equation of a parabola given sufficient graphical features
- Use the fact that two quadratic functions are equal for all values of x if and only if the corresponding coefficients are equal to solve related problems
- Model and solve practical problems involving quadratic functions and justify conclusions in the context of the problem
- Recognise that solving $f(x) = k$ for some constant k corresponds to finding the x-coordinate(s) of the intersection of the graphs $y = f(x)$ and $y = k$
- Solve problems by finding the solution to simultaneous equations involving a linear and a quadratic function, or two quadratic functions, both algebraically and graphically
- Solve quadratic inequalities
- Recognise and graph cubic functions of the form $f(x) = kx^3$ and $f(x) = k(x-a)(x-b)(x-c)$, where a, b, c and k are constants and $k \neq 0$

2.4 Exponential functions

- Graph the exponential functions $y = k(a^x)$ and $y = k(a^{-x})$ for constants a and k where $a > 0$, $a \neq 1$ and $k \neq 0$, and identify its asymptote, y-intercept, domain and range
- Describe the behaviour of $y = k(a^x)$ and $y = k(a^{-x})$ as $x \to \infty$ and $x \to -\infty$

✓ Content

☐ 2.5 Reciprocal functions

- Graph functions of the form $f(x) = \frac{k}{x}$, where k is a constant and $k \neq 0$, and identify their hyperbolic shape and their asymptotes
- Describe the behaviour of $f(x) = \frac{k}{x}$ as $x \to \infty$ and $x \to -\infty$

☐ 2.6 Direct and inverse variation

- Develop models of the form $y = kx^n$, where k is a non-zero constant, from descriptions of situations in which one quantity varies directly with another
- Develop the model $y = \frac{k}{x^n}$, where k is a non-zero constant, from descriptions of situations in which one quantity varies inversely with another
- Evaluate k in the equations $y = kx^n$ and $y = \frac{k}{x^n}$, given one pair of values for the variables, and use the resulting formula to find other values of the variables
- Analyse and solve problems involving direct and inverse variation

☐ 2.7 Circles and semicircles

- Derive the equation of a circle of radius r with centre at the origin by considering Pythagoras' theorem
- Graph circles of the form $x^2 + y^2 = r^2$ from their equations
- Determine the equation of a circle of the form $x^2 + y^2 = r^2$ given its graph
- Identify and graph the semicircles

$$y = \sqrt{r^2 - x^2} \qquad y = -\sqrt{r^2 - x^2}$$
$$x = \sqrt{r^2 - y^2} \qquad x = -\sqrt{r^2 - y^2}$$

✓ Content

☐ 2.8 Graph transformations

- Establish using graphing applications that replacing x by $-x$ in the equation of a graph corresponds to the reflection of the graph in the y-axis, and that replacing y by $-y$ corresponds to a reflection in the x-axis
- Establish using graphing applications that replacing x by $x-a$ in the equation of a graph corresponds to a horizontal translation of a graph by a, that replacing y by $y-b$ corresponds to a vertical translation by b, and recognise that the sign of a and b determines the direction of the translation
- Describe a horizontal dilation as a stretch from the y-axis and a vertical dilation as a stretch from the x-axis
- Establish using graphing applications that replacing x by $\frac{x}{k}$ in the equation of a graph corresponds to a horizontal dilation of a graph by a factor of k, that replacing y by $\frac{y}{\ell}$ corresponds to a vertical dilation by a factor of ℓ, and determine the effects on the graph when the magnitude of the dilation factor lies between 0 and 1
- Establish, using graphing applications, replacing x by $\frac{x}{k}$ and replacing y by $\frac{y}{k}$ in the equation of a graph corresponds to a dilation by a factor of k, that is, an enlargement of the graph when $k > 1$ and a reduction of the graph when $0 < k < 1$
- Apply reflections, translations and dilations to functions within the scope of the Mathematics Advanced course, excluding trigonometric functions, determine the function rule and the graph of the transformed function, the domain and range of the transformed graph, any intercepts, asymptotes and discontinuities where appropriate, and describe which transformations have been applied
- Use the principles of translations to identify the centre, radius, domain and range of graphs of circles in the form $(x-a)^2 + (y-b)^2 = r^2$, where a, b and r are constants
- Find the centre and radius of circles of the form $x^2 + y^2 + ax + by + c = 0$, where a, b and c are constants, by completing the squares
- Graph circles given their equations and find the equation of a circle from its graph
- Recognise that the order in which individual transformations are applied to functions is important

✓ Content

☐ 2.9 Piecewise-defined functions

- Interpret piecewise-defined functions, where the function is defined differently in different parts of the domain
- Graph piecewise-defined functions involving functions covered in the scope of the Mathematics Advanced course, test if they are even or odd, and determine the domain and range
- Define informally that a function is continuous at a point if the curve can be drawn through the point without lifting the pen off the paper
- Identify points where piecewise-defined functions and other functions are not continuous
- Define a discontinuity of a function informally as a point where the function is not continuous

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☐ 2.10 Absolute value functions

- Define the absolute value $|x|$ of a number x, also known as the magnitude of x, to be the distance from the origin to x on the number line
- Establish and use the piecewise definition

$$|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0 \end{cases}$$

- Show using numerical substitutions that $\sqrt{x^2} = |x|$ and use the result
- Graph the function $y = |x|$, describe its symmetry, and identify its domain and range
- Graph $y = |ax+b|$ with and without graphing applications, and identify its symmetry, domain and range
- Solve absolute value equations of the form $|ax+b| = k$ algebraically and graphically

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Reference Sheet – Mathematics Advanced **101**

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Section 1

Functions

Learning Goal(s)

Knowledge
Definitions of relations and functions

Skills
Test for relation vs functions, substitute values into function notation

Understanding
The difference between *relations* and *functions*

By the end of this section am I able to:

2.1 Introduction to functions and relations

- Describe a relation between two sets as an association between the elements of one set and the elements of the other set
- Define a real function f of a real variable x as a relation where each element x of a given set is associated with exactly one element y of the second set
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2.2 Properties of functions, relations and graphs

- Extend the definitions of domain and range to relations
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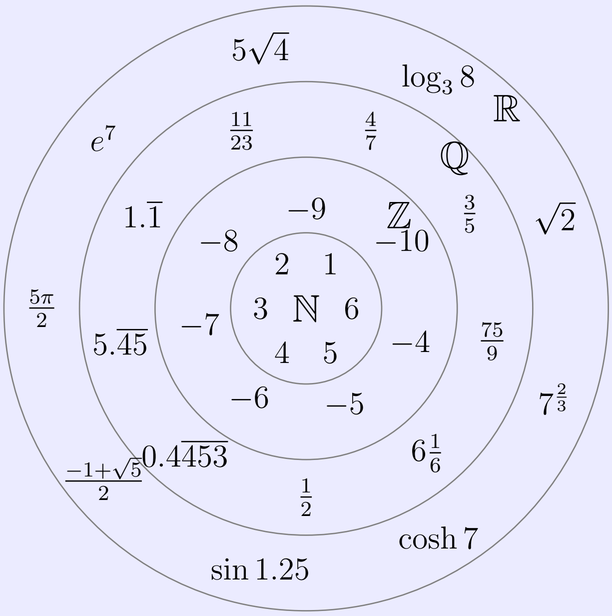
- Determine the equations of composite functions

1.1 Interval notation

Definition 1

Types of numbers



- $\mathbb{N}$: natural numbers
- $\mathbb{Z}$: integers
- $\mathbb{Q}$: rational numbers
- $\mathbb{R}$: real numbers

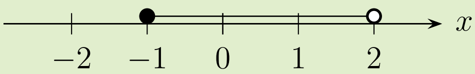
Definition 2

Set notation (read left-to-right)

- $\in$: *belongs to the* set of
- $\cap$: *and* (more formally: intersection)
- $\cup$: *or* (more formally: union)
- $A \subset B$: *A is a* subset *of B*
- $A \supseteq B$: *A is a* superset or equal to B.
- $|A|$: the magnitude (or size) of set A. Refers to the number of unique elements in the set.
- $\overline{A}$, A^c or A': the complement of set A. Everything that is not in the set A.
- $\varnothing$: the empty set. If $|A| = 0$, then $A = \varnothing$
- $[a, b]$: all real numbers from a to b inclusive.
- $(a, b]$: all real numbers from a to b, exclusive at a, inclusive at b.

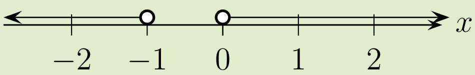
Example 1

Write in interval notation:



Example 2

Write in interval notation:



Exercises

Source These exercises are from Coroneos (1990, Set 2C, p.49-50).

1. Use braces to illustrate with inequality signs the following intervals. Illustrate each interval on the real number line.

(a) $(-3, 0]$ (b) $[0, 1]$ (c) $(0, \infty)$ (d) $[1, 3)$

2. Use parentheses "()" and brackets "[]" to denote

(a) $\{x : 1 \le x < 5\}$ (c) $\{x : x \ge 0\}$

(b) $\{x : x < -1\}$ (d) $\{x : -3 \le x \le 2\}$

3. Which of the operations ($\cup$ or $\cap$) will make the first two intervals, the third?

(a) $(1, 3)$, $(2, 4)$, $(2, 3)$ (b) $[-1, 2)$, $[0, 4)$, $[-1, 4)$

4. Illustrate on the number line:

(a) $(-1, 0]$ (b) $[-1, 2]$ (c) $[-1, 2)$ (d) $(-1, 2)$

5. State whether $A = B$, $A \subset B$ or $B \subset A$ in the following:

(a) $A = \{x : 2 < x < 5\}$, $B = (0, 6)$ (c) $A = (0, 1]$, $B = [0, 1]$

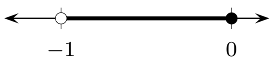
(b) $A = [2, 3]$, $B = (2, 3)$ (d) $A = [0, 1]$, $B = \{x : 0 \le x \le 1\}$

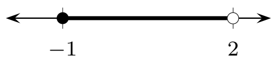
Answers

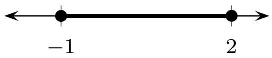
1. (a) $\{x : -3 < x \le 0\}$ (b) $\{x : 0 \le x \le 1\}$ (c) $\{x : x > 0\}$ (d) $\{x : 1 \le x < 3\}$

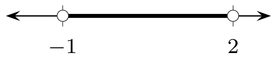
2. (a) $[1, 5)$ (b) $(-\infty, -1)$ (c) $[0, \infty)$ (d) $[-3, 2]$

3. (a) $\cap$ (b) $\cup$

4. (a)


(c)


(b)


(d)


5. (a) $A \subset B$ (b) $B \subset A$ (c) $A \subset B$ (d) $A = B$

Further exercises

Ⓐ **Ex 10C** (Pender et al., 2019a)
- Q1-14

(X1) **Ex 12C** (Pender et al., 2019b)
- Q1-17

1.2 Relations and functions

Definition 3

A **relation** is a ....rule.... that maps ....elements.... from the ....domain.... to the ....range.... .

- The ....domain.... is the set of all possible x values.
- The ....range.... is the set of all possible y values.

Laws/Results

(X1) Four types of relations:

- One to one
- One to many
- Many to one
- Many to many

Definition 4

Functions are relations in which each ....input.... (x *-value*) maps to **only one** ....output.... (y *-value*).

Laws/Results

Only ....one.... ....to.... ....one.... and ....many.... ....to.... ....one.... relations are *functions*.

Definition 5

Vertical line test If a vertical line ....intersects........ or ......touches...... a graph at more than one point, then the graph is ...not... a function.

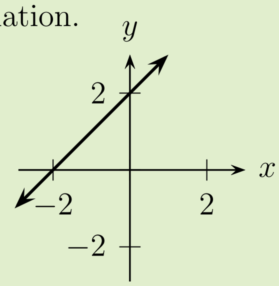
Important note

If a graph is available, use the (informal) *vertical line test.*

Example 3

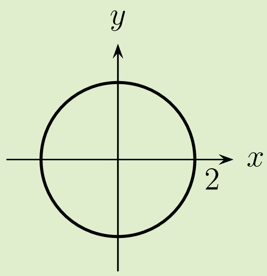
State whether the following are functions or simply relations. Also, classify the type of relation.

(a)



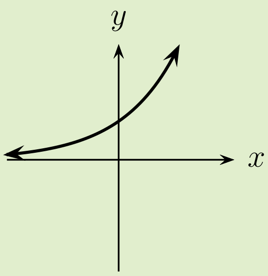
- Function/not a function
- Type: ....One to one........
- Domain: ....$D = \{x : x \in \mathbb{R}\}$........
- Range: ....$R = \{y : y \in \mathbb{R}\}$........

(b)



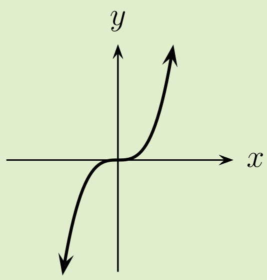
- Function/not a function
- Type: ....Many to many........
- Domain: ....$D = \{x : -2 \le x \le 2\}$........
- Range: ....$R = \{y : -2 \le y \le 2\}$........

(c)



- Function/not a function
- Type: ....One to one........
- Domain: ....$D = \{x : x \in \mathbb{R}\}$........
- Range: ....$R = \{y : y > 0\}$........

(d)

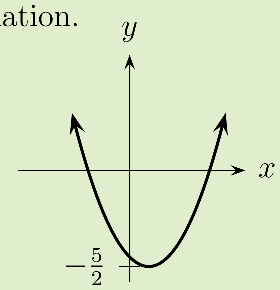


- Function/not a function
- Type: ....One to one........
- Domain: ....$D = \{x : x \in \mathbb{R}\}$........
- Range: ....$R = \{y : y \in \mathbb{R}\}$........

Example 4

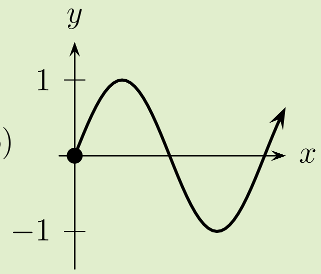
State whether the following are functions or simply relations. Also, classify the type of relation.

(a)



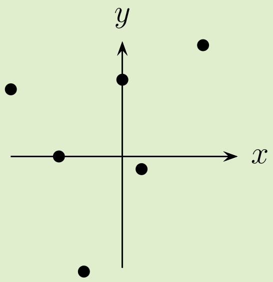
- Function/not a function
- Type: Many to one
- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $R = \{y : y \geq -\frac{5}{2}\}$

(b)



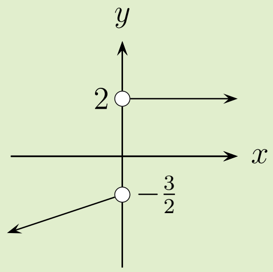
- Function/not a function
- Type: Many to one
- Domain: $D = \{x : x \geq 0\}$
- Range: $R = \{y : -1 \leq y \leq 1\}$

(c)



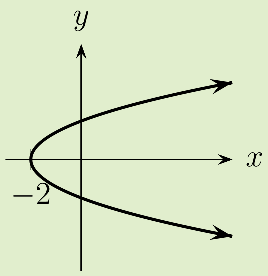
- Function/not a function
- Type: One to one

(d)



- Function/not a function
- Type: Many to one
- Domain: $D = \{x : x \in \mathbb{R}\backslash\{0\}\}$
- Range: $R = \{y : y < -\frac{3}{2}$ or $y = 2\}$

(e)



- Function/not a function
- Type: One to many
- Domain: $D = \{x : x \geq -2\}$
- Range: $R = \{y : y \in \mathbb{R}\}$

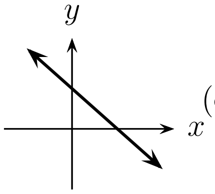
Additional questions

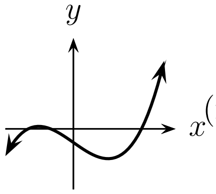
1. Write down the domain and range of each relation:

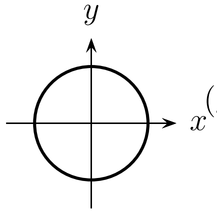
(a) $\{(1,3),(3,0),(5,4)\}$ (b) $\{(1,1),(2,1),(3,1)\}$

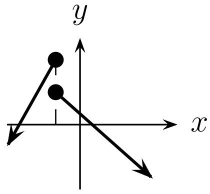
2. State whether each relation is a *function* or *not a function*:

(a) $\{(1,1),(2,2),(3,3)\}$ (b) $\{(1,2),(2,3),(1,4)\}$ (c) $\{(3,2),(4,3),(5,3)\}$

(d)


(e)


(f)


(g)


(h) $y = 2x$ (i) $x = y^2$ (j) $y = 2$ (k) $x = 1$

Answers

1. (a) $D = \{1,3,5\}$, $R = \{0,3,4\}$. (b) $D = \{1,2,3\}$, $R = \{1\}$ **2. Functions:** (a), (c), (d), (e), (h), (j) **Relations only:** (b), (f), (g), (i), (k)

Further exercises

Ⓐ **Ex 3B** (Pender et al., 2019a)
- Q1, 2

Ⓧ1 **Ex 3B** (Pender et al., 2019b)
- Q1, 2

1.3 Function notation

Algebraic rule	Name	Placeholder
Dependent variable		Independent variable

$$f(x) = x^2 - 4x \qquad g(\spadesuit) = \spadesuit^2 - 4\spadesuit$$

Example 5

For the function $f(x) = x^2 + 4x + 1$,

(a) Find $f(1)$

(b) Find $f(-3)$

(c) Find $f(k)$

(d) Find $f(k+1)$

(e) Find $f(3-k)$

(f) Sketch the function

Further exercises

Ⓐ **Ex 3A** (Pender et al., 2019a)

- Q4-5, 7-9, 11-17

Ⓧ1 **Ex 3A** (Pender et al., 2019b)

- Q3-4, 6, 8, 9, 10, 12-17, 19

1.3.1 Composition of functions

Definition 6

If $f(x) = x^2$, then f maps x to x^2, i.e.

$$f : x \mapsto x^2$$

Definition 7

If there are two mappings h and g, then their *composition*

- $h \circ g$: maps g, followed by h
- $g \circ h$: maps h, followed by g

(Draw function chain diagrams and regular function notation to illustrate)

Laws/Results

If f and g are functions and the composite function $f\big(g(x)\big)$ exists:

- The *outer* function is f, the *inner* function is g.
- The ......range...... of the ......inner...... function must be a subset of the ......domain...... of the ......outer...... function.
  - The ......domain...... of the ......inner...... function may need to be restricted
- The ......domain...... of the ......inner...... function (or its restricted version) is the ......domain...... of the composite function.

Example 6

For the functions $f(x) = 7x^2 + 3$ and $g(x) = (2x+5)^2$, draw a 'function machine' depicting the input/outputs, and also evaluate:

(a) $f\big(g(x)\big)$.

(b) $g\big(f(x)\big)$.

Example 7

[Ex 13:03] (McSeveney, Conway, & Wilkes, 1986) If $h : x \mapsto x^2$ and $g : x \mapsto x + 3$, then find

(a) $h \circ g(1)$

(b) $g \circ h(-2)$

(c) $g \circ h(a)$

(d) $h \circ g(a)$

Answer: (a) 16 (b) 7 (c) $a^2 + 3$ (d) $(a+3)^2$

Example 8

[2021 Independent Adv Trial] If $f(x) = \sqrt{x}$ and $g(x) = 4 - x^2$, what is the domain of the function $f\Big(g(x)\Big)$?

(A) $[-2, 2]$ (C) $(-\infty, 2] \cup [2, \infty)$

(B) $[0, \infty)$ (D) $(-\infty, \infty)$

Example 9

[2023 CSSA Adv Trial Q5] Let $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{x-1}$.

What is the domain and range for both $f\Big(g(x)\Big)$ and $g\Big(f(x)\Big)$?

(A) **Domain:** $[-1, 1]$ **Range:** $(-\infty, \infty)$

(B) **Domain:** $(0, \infty)$ **Range:** $(-\infty, 0) \cup (-, \infty)$

(C) **Domain:** $[0, \infty)$ **Range:** $(0, \infty)$

(D) **Domain:** $(1, \infty)$ **Range:** $(0, \infty)$

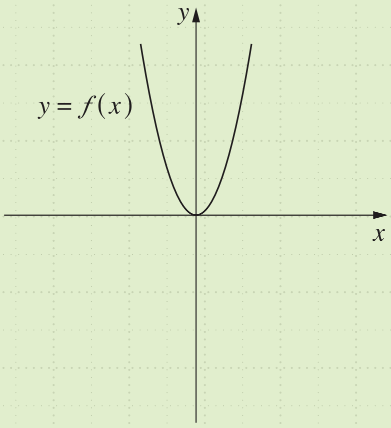
Example 10

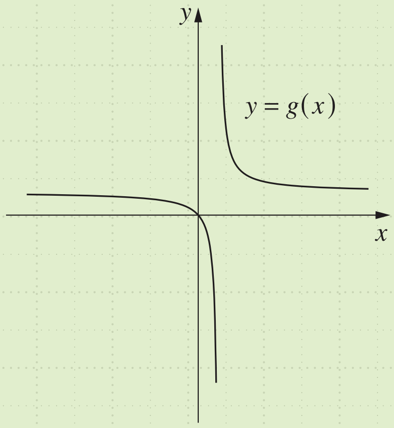
[2025 NBHS Year 11 Ext 1 Yearly Examination] (3 marks)

Given $f(x) = \dfrac{x+2}{3x-1}$ and $f\Big(g(x)\Big) = \dfrac{3x+2}{2x-5}$, find $g(x)$. **Answer:** $g(x) = \frac{3x+11}{7x-8}$

Example 11

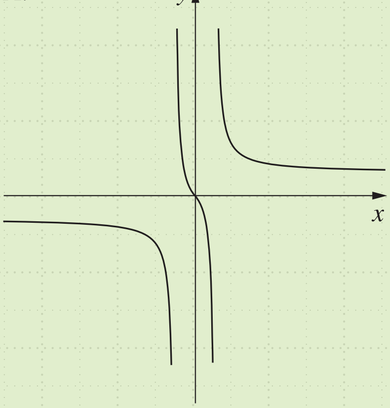
[2021 Adv HSC Q10] The graphs of $y = f(x)$ and $y = g(x)$ are shown.



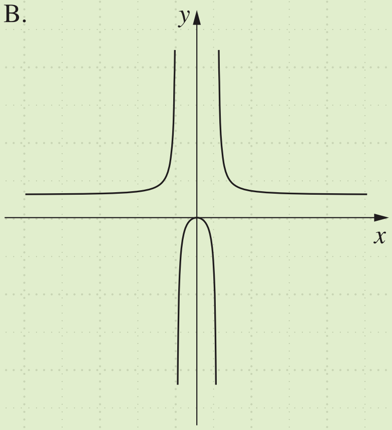


Which of the following best represents $g\Big(f(x)\Big)$?

A.

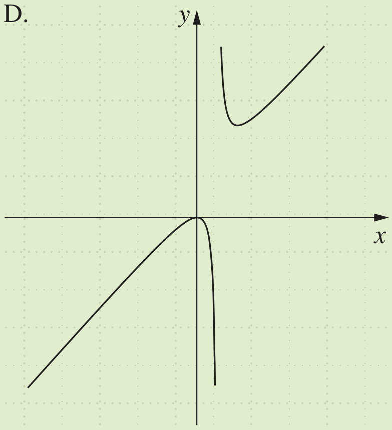


B.



C.

D.



Further exercises

Ⓐ **Ex 4E** (Pender et al., 2019a)

- Q1-15

(X1) **Ex 4E** (Pender et al., 2019b)

- Q1-17

Additional questions

1. If $f(x) = 2x^2 - 1$, evaluate:

(a) $f(0)$ (c) $f(-2)$ (e) $f(a)$

(b) $f(3)$ (d) $3f(1) - (f(2))^2$ (f) $f(x+1)$

2. If $g(x) = 2^x + \dfrac{1}{x-1}$,

(a) Evaluate: i. $g(0)$ ii. $g(-4)$ as a fraction.

(b) For what value of x does $g(x)$ not exist?

3. If $f(x) = x^2 + 1$ and $g(x) = 2x - 3$, evaluate and simplify:

(a) $f[g(0)]$ (b) $g[f(0)]$ (c) $f[g(x)]$ (d) $g[f(x)]$

4. Evaluate $h(3)$ if

(a) $h(x) = 5 - x$ (d) $h(x) = 6$

(b) $h(x) = \sqrt{25 - x^2}$ (e) $h(x) = |x|$

(c) $h(x) = 6x$ (f) $h(x) = |2x - 1| - |3 - 5x|$

5. If $f(x) = 3x - 2$, solve

(a) $f(x) = 0.$ (b) $f(x) = x + 1$ (c) $f(x) < 1$

6. If $P(x) = x^2 - x - 2$, solve

(a) $P(x) = 0$ (b) $P(x) = 4$

7. If $A(x) = 2x^2 + 7x - 4$ and $B(x) = 2x - 1$, solve $A(x) = B(x)$.

8. If $f(x) = 2^x$, for what value of x does $f(x) = \dfrac{1}{4}$?

9. If $f(x) = x^2 + x$, simplify fully: $\dfrac{f(x+h) - f(x)}{h}$.

10. If $g(x) = x^2 - x - 1$, for what values of x does $g(x) = g(-x)$?

11. If $f(x) = ax + b$, and $f(3) = 2$, $f(4) = 4$, find a and b.

12. If $H : x \mapsto 2x - 1$ and $G : x \mapsto x^2$, find:

(a) $H \circ G(1)$ (b) $G \circ H(1)$ (c) $H \circ G(2)$ (d) $G \circ H(2)$

13. If $m : x \mapsto x - 3$ and $n : x \mapsto x^2$, find:

(a) $m \circ n(x)$ (b) $n \circ m(x)$

(c) the value of x such that $m \circ n(x) = n \circ m(x)$

14. If $h(x) = \dfrac{1}{x}$, $x \neq 0$ and $g(x) = x - 2$, find a value of x for which $h \circ g(x)$ does not exist.

15. **[2021 CSSA Adv Trial Q14]** Consider the function $f(x) = ax + b$, where a and b are constants. The function satisfies $f(-3) = 9$ and $f(5) = -7$.

(a) Find the values of a and b. **2**

(b) Solve the equation $f\Big(f(x)\Big) = 0$. **1**

The following are sourced from Swale et al. (2019, Ex 3.2), modified to suit NSW syllabuses.

16. If $f(x) = (x-1)(x+3)$ and $g(x) = x^2$, investigate whether the composition function $f\Big(g(x)\Big)$ and $g\Big(f(x)\Big)$ exist. If they do, form the rule for the composite function and state the domain.

17. If $f(x) = 2x - 1$ and $g(x) = \dfrac{1}{x-2}$, investigate whether the composition function $f\Big(g(x)\Big)$ and $g\Big(f(x)\Big)$ exist. If they do, form the rule for the composite function and state the domain.

18. For the functions $f(x) = x^2 + 1$, $g(x) = \sqrt{x}$ and $h(x) = \dfrac{1}{x}$, determine whether the following composite functions are defined or undefined. If the composite function exists, identify its domain.

(a) $f \circ g(x)$ (b) $g\Big(f(x)\Big)$ (c) $h\Big(g(x)\Big)$ (d) $h \circ f(x)$

19. For the functions $f(x) = x^2$, $g(x) = \sqrt{x}$ and $h(x) = -\dfrac{1}{x}$, determine whether the following composite functions are defined or undefined. If the composite function exists, identify its domain.

(a) $f \circ g(x)$ (b) $g\Big(f(x)\Big)$ (c) $h \circ f(x)$ (d) $g\Big(h(x)\Big)$

20. The functions f and g are defined by:

$$f(x) = x^2 + 1 \quad g(x) = \sqrt{x+2}$$

Show that $f\Big(g(x)\Big)$ exists and find the rule for $f\Big(g(x)\Big)$, stating its domain and range.

21. Given the following information:

$$f(x) = \frac{1}{x} \quad D_f = (0, \infty)$$
$$g(x) = \frac{1}{x^2} \quad D_g = R_f$$

(a) Prove that $g\Big(f(x)\Big)$ exists.

(b) Find $g\Big(f(x)\Big)$ and state its domain and range.

(c) Sketch the graph of $y = g\Big(f(x)\Big)$.

22. For the functions $f(x) = \sqrt{x+3}$ and $g(x) = 2x - 5$,

(a) State why $f\Big(g(x)\Big)$ is not defined.

(b) Restrict the domain of g to form a new function, $h(x)$, such that $f\Big(h(x)\Big)$ is defined.

(c) Find $f\Big(h(x)\Big)$

23. For the functions $f(x) = x^2$ and $g(x) = \dfrac{1}{x-4}$

(a) State why $g\Big(f(x)\Big)$ is not defined.

(b) Restrict the domain of f to form a new function, $h(x)$, such that $g\Big(h(x)\Big)$ is defined.

(c) Find $g\Big(h(x)\Big)$

24. If $g(x) = \dfrac{1}{(x-3)^2} - 2$ and $f(x) = \sqrt{x}$,

(a) Prove that $f\Big(g(x)\Big)$ is not defined.

(b) Restrict the domain of g to form a new function, $g_1(x)$, such that $f\Big(g_1(x)\Big)$ exists.

25. For the equations $f(x) = \sqrt{x-4}$ and $g(x) = x^2 - 2$,

(a) Prove that $g\Big(f(x)\Big)$ is defined.

(b) Find the rule for $g\Big(f(x)\Big)$ and state the domain.

(c) Sketch the graph of $y = g\Big(f(x)\Big)$

(d) Prove that $f\Big(g(x)\Big)$ is not defined.

(e) Restrict the domain of g to obtain a function $g_1(x)$ such that $f\Big(g_1(x)\Big)$ exists.

(f) Find $f\Big(g_1(x)\Big)$.

26. Given

$$f(x) = -\sqrt{x} + k \quad D_f = [1, \infty)$$
$$g(x) = x^2 + k \quad D_g = (-\infty, 2]$$

where $k \in \mathbb{R}^+$, find the value(s) of k such that both $f\Big(g(x)\Big)$ and $g\Big(f(x)\Big)$ are defined.

Answers

1. (a) -1 (b) 17 (c) 7 (d) -46 (e) $2a^2 - 1$ (f) $2x^2 + 4x + 1$ **2.** (a) i. 0 ii. $-\frac{11}{80}$ (b) $x = 1$ **3.** (a) 10 (b) -1 (c) $4x^2 - 12x + 10$ (d) $2x^2 - 1$ **4.** (a) 2 (b) 4 (c) 18 (d) 6 (e) 3 (f) -7 **5.** (a) $x = \frac{2}{3}$ (b) $x = \frac{3}{2}$ (c) $x < 1$ **6.** (a) $x = 2, -1$ (b) $x = -2, 3$ **7.** $x = \frac{1}{2}, -3$ **8.** $x = -2$ **9.** $2x + h + 1$ **10.** $x = 0$ **11.** $a = 2$, $b = -4$ **12.** (a) 1 (b) 1 (c) 7 (d) 9 **13.** (a) $x^2 - 3$ (b) $(x-3)^2$ (c) 2 **14.** $x = 2$ **15.** (a) $a = -2$, $b = 3$ (b) $x = \frac{3}{4}$ **16.** $f\Big(g(x)\Big) = (x-1)(x+1)\left(x^2+3\right)$, $D = \mathbb{R}$; $g\Big(f(x)\Big) = (x-1)^2(x+3)^2$, $D = \mathbb{R}$. **17.** $f\Big(g(x)\Big) = \frac{2}{x-2} - 1$, $D = \mathbb{R}\backslash\{2\}$; $g\Big(f(x)\Big)$ does not exist. **18.** (a) $f \circ g(x) = x + 1$, $D = [0, \infty)$ (b) $g\Big(f(x)\Big) = \sqrt{x^2+1}$, $D = \mathbb{R}$ (c) $h\Big(g(x)\Big)$ not defined (d) $h \circ f(x) = \frac{1}{x^2+1}$, $D = \mathbb{R}$ **19.** (a) $f \circ g(x) = x$, $D = [0, \infty)$ (b) $g\Big(f(x)\Big) = |x|$, $D = \mathbb{R}$ (c) $h \circ f(x)$ not defined (d) $g\Big(f(x)\Big)$ not defined **20.** $f\Big(g(x)\Big) = x + 3$, $D = [-2, \infty)$, $R = [0, \infty)$ **21.** (a) $[0, \infty) \subseteq \mathbb{R}\backslash\{0\}$ (b) $g\Big(f(x)\Big) = x^2$, $D = (0, \infty)$, $R = (0, \infty)$ **22.** (a) Range of g is not a subset, or equal set to the domain of f: $\mathbb{R} \not\subseteq [-3, \infty)$ (b) $h(x) = 2x - 5$, $x \in [1, \infty)$ (c) $f\Big(h(x)\Big) = \sqrt{2x-2}$, $x \in [1, \infty)$ **23.** (a) Range of f is not a subset, or equal set to the domain of g: $[0, \infty) \not\subseteq \mathbb{R}\backslash\{4\}$ (b) $h(x) = x^2$, $x \in \mathbb{R}\backslash\{-2, 2\}$ (c) $g\Big(h(x)\Big) = \frac{1}{x^2-4}$, $x \in \mathbb{R}\backslash\{-2, 2\}$ **24.** (a) Range of g is not a subset, or equal set to the domain of f: $(-2, \infty) \not\subseteq [0, \infty)$ (b) $g_(x) = \frac{1}{(x-3)^2} - 2$, $x \in \left(-\infty, -\frac{1}{\sqrt{2}} + 3\right] \cup \left[\frac{1}{\sqrt{2}} + 3, \infty\right)$ (c) $g\Big(h(x)\Big) = \frac{1}{x^2-4}$, $x \in \mathbb{R}\backslash\{-2, 2\}$ **25.** (a) $[0, \infty) \subseteq \mathbb{R}$. Hence, $g\Big(f(x)\Big)$ is defined. (b) $g\Big(f(x)\Big) = x - 6$, $x \in [4, \infty)$. (c) Sketch (d) $[-2, \infty) \not\subseteq [4, \infty)$ (e) $g_1(x) = x^2 - 2$, $x \in \left(-\infty, -\sqrt{6}\right] \cup \left[\sqrt{6}, \infty\right)$ (f) $f\left(g_1(x)\right) = \sqrt{x^2-6}$, $x \in \left(-\infty, -\sqrt{6}\right] \cup \left[\sqrt{6}, \infty\right)$. **26.** $k \in [1, 3]$

1.4 Basic functions/relations and their natural domain/range

Definition 8

- **Domain** The set of elements that a relation maps from.

  ➜ The natural **domain** of a function is the maximum set of values for which the function is defined.

- **Range** The set of elements that a relation can map to.

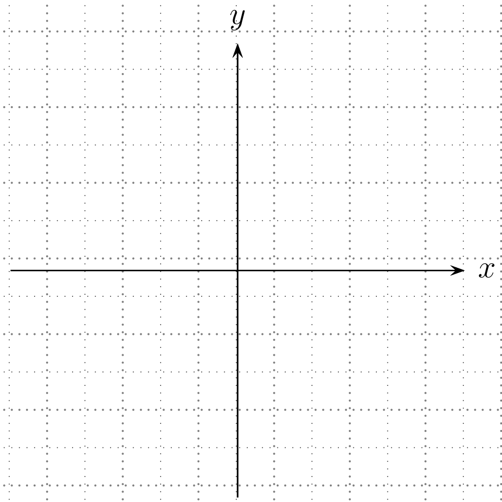
Important note

What is the difference between the domain and natural domain?

Important note

The following curves *must* be known. Sketch the following graphs in their most basic form, and state their natural domain & range using interval notation:

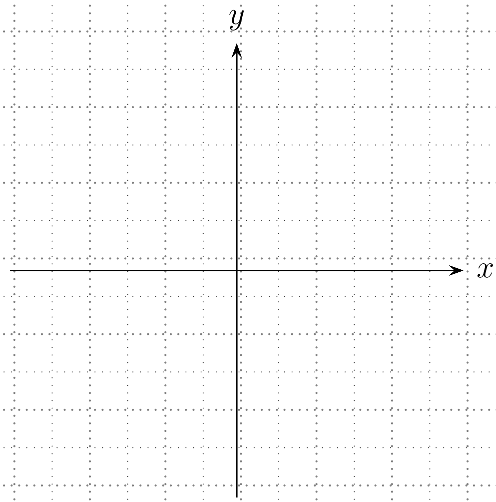
- Straight line $y = x$



Natural domain: ....

Range: ....

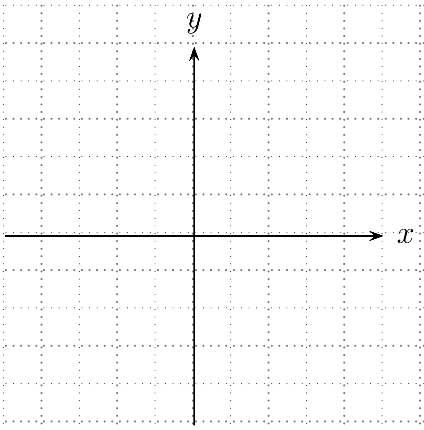
- Parabola $y = x^2$



Natural domain: ....

Range: ....

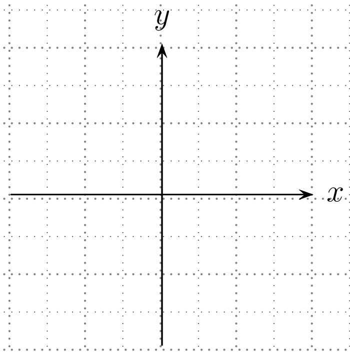
- Basic cubic $y = x^3$



Natural domain: ....

Range: ....

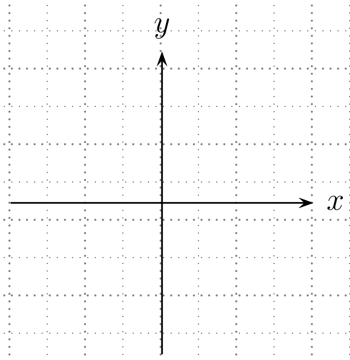
- Hyperbola $y = \frac{1}{x}$



Natural domain: ....

Range: ....

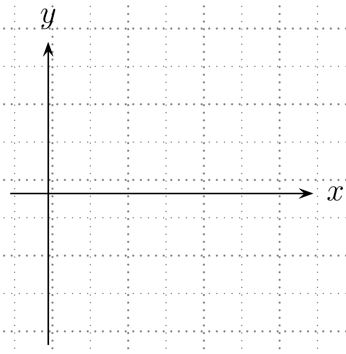
- Circle $x^2 + y^2 = r^2$



Natural domain: ....

Range: ....

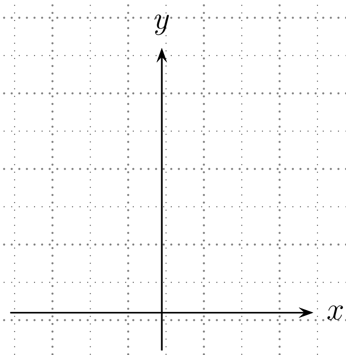
- Logarithmic $y = \log x$



Natural domain: ....

Range: ....

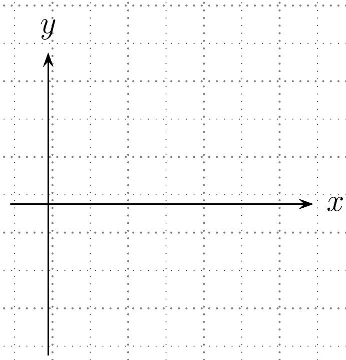
- Exponential $y = a^x$



Natural domain: ....

Range: ....

- Trigonometric $y = \sin x$



Natural domain: ....

Range: ....

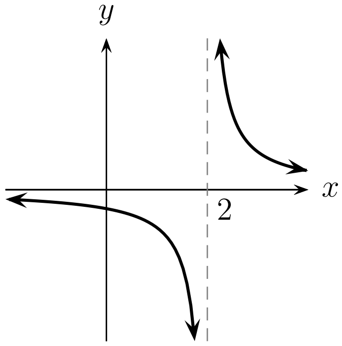
Additional questions

1. State the values of x which are *excluded* from the domain of the following:

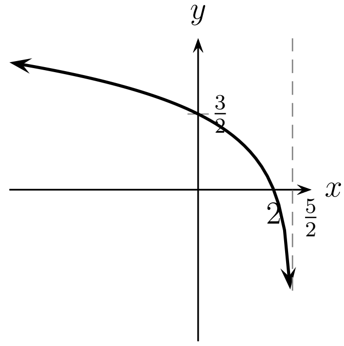
(a) $y = \dfrac{1}{x}$ (b) $y = \sqrt{x}$ (c) $y = \sqrt{9 - x^2}$

2. State the exclusions from the domain shown by the following graphs.

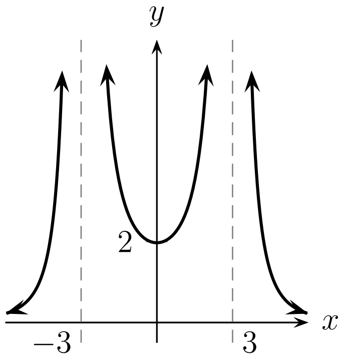
(a)



(c)



(b)



3. State the natural domain of each of the following:

(a) $y = \sqrt{x - 6}$

(b) $y = \dfrac{1}{x+2}$

(c) $y = \sqrt{3x + 1}$

(d) $y = \dfrac{2}{x^2 - 4}$

(e) $y = \sqrt{4 - x^2}$

(f) $y = \dfrac{1}{x+2} + \dfrac{1}{x-3}$

(g) $y = \sqrt{x - 6} + \sqrt{x + 3}$

(h) $y = \dfrac{3}{\sqrt{x - 4}}$

(i) $y = \dfrac{5}{\sqrt{16 - x^2}}$

(j) $y = \dfrac{x}{(x+9)(2x-3)}$

(k) $y = \sqrt{x^2 - 9}$

(l) $y = \dfrac{3}{\sqrt{x^2 - 1}}$

4. For the following functions and relations:

i. Draw neat sketches of the following functions and relations, showing all important features of the graphs

ii. State the natural domain and range using *interval notation*.

(a) $x = 3$

(b) $y = 2x - 4$

(c) $3x + 4y - 12 = 0$

(d) $y = -x^2$

(e) $y = 2x^2 + 1$

(f) $y = (x - 2)^2$

(g) $y = (x - 2)(x - 4)$

(h) $y = x^2 - x - 6$

(i) $y = 8 + 2x - x^2$

(j) $y = (x + 3)^2 + 1$

(k) $y = 4 - (x - 5)^2$

(l) $y = -x^3$

(m) $y = 3 + 2x^3$

(n) $y = x^8$

(o) $y = -x^9$

(p) $x^2 + y^2 = 16$

(q) $x^2 + (y - 3)^2 = 4$

(r) $(x - 1)^2 + (y + 2)^2 = 1$

(s) $x^2 + y^2 + 8x - 6y - 11 = 0$

(t) $x^2 + y^2 - 2x = 0$

(u) $9x^2 + 9y^2 + 9x + 6y + 1 = 0$

(v) $y = 3^x$

(w) $y = -4^{-x}$

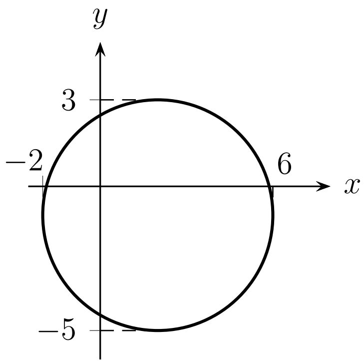
(x) $y = 1 - 2^{-x}$

(y) $y = -\dfrac{3}{x}$

(z) $xy = 8$

5. A standard hyperbola has the x and y axes as its asymptotes, and passes through the point $(3, -2)$. Find the equation.

6. Find the equation of this circle in general form. The numbers on the axes are the extremities of the circle, not the intercepts with the axes.



Further exercises

Ⓐ **Ex 3B** (Pender et al., 2019a)
- Q3, 5, 6, 8, 14

Ⓐ **Ex 3D**
- Q6-9

Ⓐ **Ex 3E**
- Q5

Ⓐ **Ex 3F**
- Q6

X1 **Ex 3B** (Pender et al., 2019b)
- Q3-5, 8, 13

X1 **Ex 3D**
- Q7-9

X1 **Ex 3E**
- Q9

X1 **Ex 3F**
- Q3

Answers

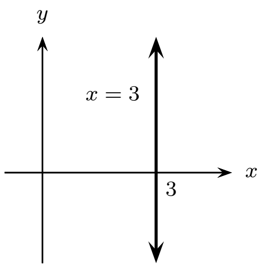
1. (a) $x = 0$
(b) $x < 0$
(c) $x < -3$ or $x > 3$

2. (a) $x = 2$
(b) $x = \pm 3$
(c) $x \geq \frac{5}{2}$

3. (a) $D = \{x : x \in [6, \infty)\}$
(b) $D = \{x : x \in \mathbb{R} \backslash \{-2\}\}$
(c) $D = \left\{x : x \in \left[-\frac{1}{3}, \infty\right)\right\}$
(d) $D = \{x : x \in \mathbb{R} \backslash \{-2, 2\}\}$
(e) $D = \{x : x \in [-2, 2]\}$
(f) $D = \{x : x \in \mathbb{R} \backslash \{-2, 3\}\}$
(g) $D = \{x : x \in [6, \infty)\}$
(h) $D = \{x : x \in (4, \infty)\}$
(i) $D = \{x : x \in (-4, 4)\}$
(j) $D = \left\{x : x \in \mathbb{R} \backslash \left\{-9, \frac{3}{2}\right\}\right\}$
(k) $D = \{x : x \in]-3, 3[\}$
(l) $D = \{x : x \in)-1, 1(\}$

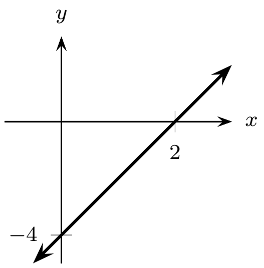
4. (a) $x = 3$

$$D = \{x : x = 3\}$$
$$R = \{y : y \in \mathbb{R}\}$$



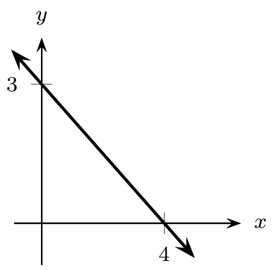
(b) $y = 2x - 4$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in \mathbb{R}\}$$



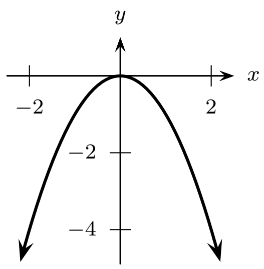
(c) $3x + 4y - 12 = 0$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in \mathbb{R}\}$$



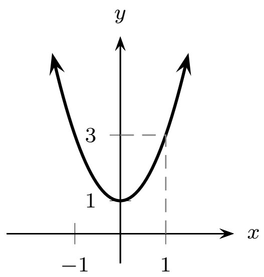
(d) $y = -x^2$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (-\infty, 0]\}$$



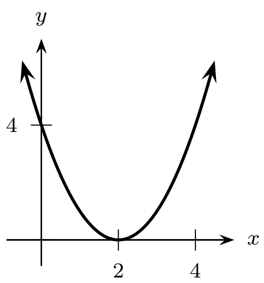
(e) $y = 2x^2 + 1$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in [1, \infty)\}$$



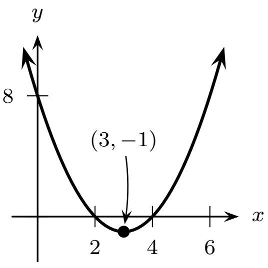
(f) $y = (x - 2)^2$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in [0, \infty)\}$$



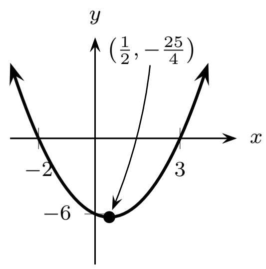
(g) $y = (x - 2)(x - 4)$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in [-1, \infty)\}$$



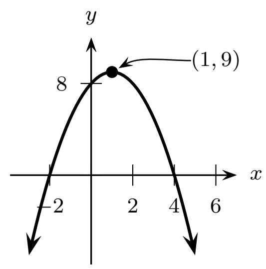
(h) $y = x^2 - x - 6$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \left\{y : y \in \left[-\frac{25}{4}, \infty\right)\right\}$$



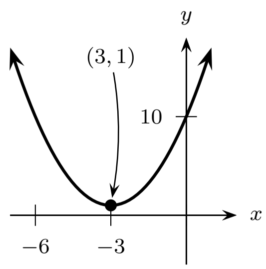
(i) $y = 8 + 2x - x^2$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (-\infty, 9]\}$$



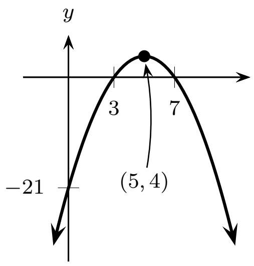
(j) $y = (x + 3)^2 + 1$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in [1, \infty)\}$$



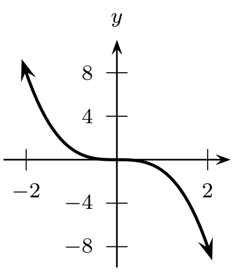
(k) $y = 4 - (x-5)^2$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (-\infty, 4]\}$$



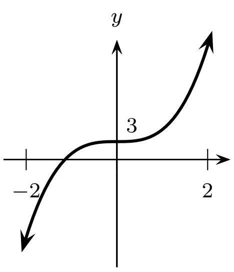
(l) $y = -x^3$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in \mathbb{R}\}$$



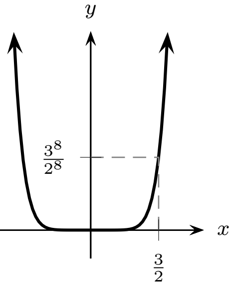
(m) $y = 3 + 2x^3$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in \mathbb{R}\}$$



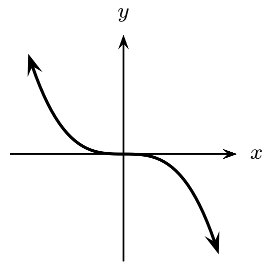
(n) $y = x^8$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in [0, \infty)\}$$



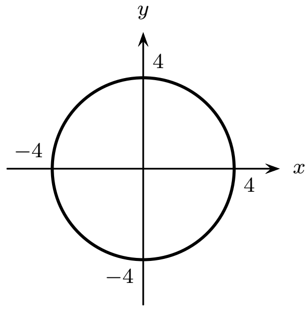
(o) $y = -x^9$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in \mathbb{R}\}$$



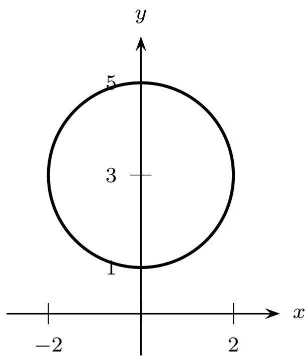
(p) $x^2 + y^2 = 16$

$$D = \{x : x \in [-4, 4]\}$$
$$R = \{y : y \in [-4, 4]\}$$



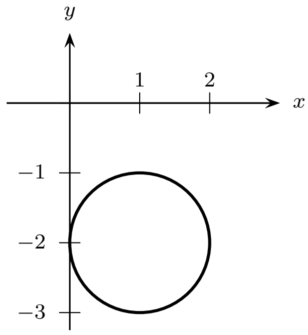
(q) $x^2 + (y-3)^2 = 4$

$$D = \{x : x \in [-2, 2]\}$$
$$R = \{y : y \in [1, 5]\}$$



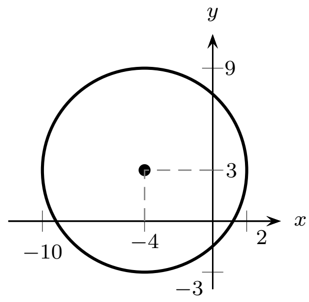
(r) $(x-1)^2 + (y+2)^2 = 1$

$$D = \{x : x \in [0, 2]\}$$
$$R = \{y : y \in [-3, -1]\}$$



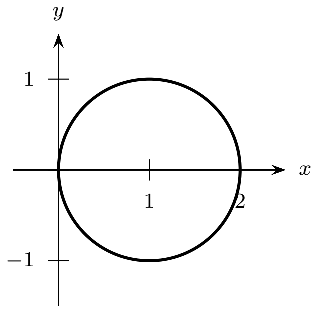
(s) $x^2 + y^2 + 8x - 6y - 11 = 0$

$$D = \{x : x \in [-10, 2]\}$$
$$R = \{y : y \in [-3, 9]\}$$



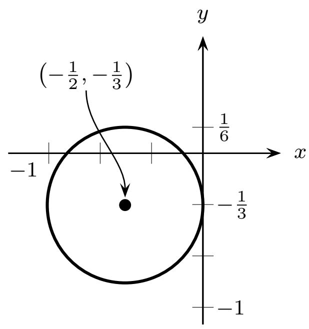
(t) $x^2 + y^2 - 2x = 0$

$$D = \{x : x \in [0, 2]\}$$
$$R = \{y : y \in [-1, 1]\}$$



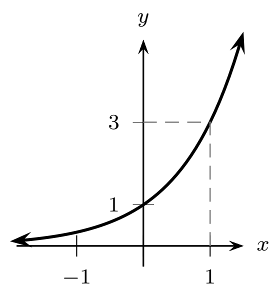
(u) $9x^2 + 9y^2 + 9x + 6y + 1 = 0$

$$D = \{x : x \in [-1, 0]\}$$
$$R = \left\{y : y \in \left[-\tfrac{5}{6}, \tfrac{1}{6}\right]\right\}$$



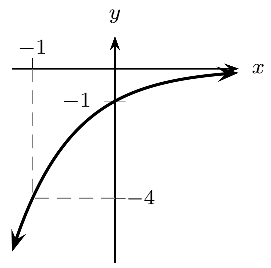
(v) $y = 3^x$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (0, \infty)\}$$



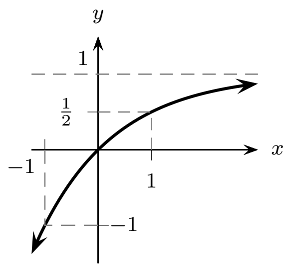
(w) $y = -4^{-x}$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (-\infty, 0)\}$$



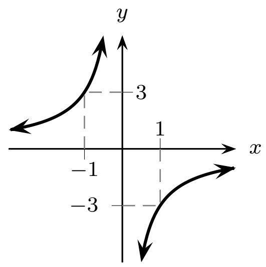
(x) $y = 1 - 2^{-x}$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (-\infty, 1)\}$$



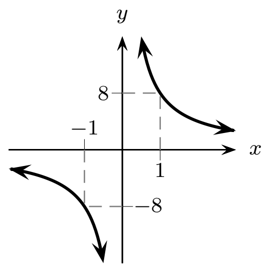
(y) $y = -\dfrac{3}{x}$

$$D = \{x : x \neq 0\}$$
$$R = \{y : y \neq 0\}$$



(z) $xy = 8$

$$D = \{x : x \neq 0\}$$
$$R = \{y : y \neq 0\}$$



5. $xy = -6$

6. $x^2 + y^2 - 4x + 2y - 11 = 0$

1.5 Curve transformations

Learning Goal(s)

Knowledge
How curves are transformed

Skills
Relating the algebra of curve transformation to the graphical representation

Understanding
The differences in algebra when transforming vertically versus horizontally

By the end of this section am I able to:

2.8 Graph transformations

- Establish using graphing applications that replacing x by $-x$ in the equation of a graph corresponds to the reflection of the graph in the y-axis, and that replacing y by $-y$ corresponds to a reflection in the x-axis
- Establish using graphing applications that replacing x by $x - a$ in the equation of a graph corresponds to a horizontal translation of a graph by a, that replacing y by $y - b$ corresponds to a vertical translation by b, and recognise that the sign of a and b determines the direction of the translation
- Describe a horizontal dilation as a stretch from the y-axis and a vertical dilation as a stretch from the x-axis
- Establish using graphing applications that replacing x by $\frac{x}{k}$ in the equation of a graph corresponds to a horizontal dilation of a graph by a factor of k, that replacing y by $\frac{y}{\ell}$ corresponds to a vertical dilation by a factor of ℓ, and determine the effects on the graph when the magnitude of the dilation factor lies between 0 and 1
- Establish, using graphing applications, replacing x by $\frac{x}{k}$ and replacing y by $\frac{y}{k}$ in the equation of a graph corresponds to a dilation by a factor of k, that is, an enlargement of the graph when $k > 1$ and a reduction of the graph when $0 < k < 1$
- Apply reflections, translations and dilations to functions within the scope of the Mathematics Advanced course, excluding trigonometric functions, determine the function rule and the graph of the transformed function, the domain and range of the transformed graph, any intercepts, asymptotes and discontinuities where appropriate, and describe which transformations have been applied
- Use the principles of translations to identify the centre, radius, domain and range of graphs of circles in the form $(x - a)^2 + (y - b)^2 = r^2$, where a, b and r are constants
- Find the centre and radius of circles of the form $x^2 + y^2 + ax + by + c = 0$, where a, b and c are constants, by completing the squares
- Graph circles given their equations and find the equation of a circle from its graph
- Recognise that the order in which individual transformations are applied to functions is important

Theorem 1

If $a > 0$,

Translations

- $f(x-a)$ shifts right by a units.
- $f(x+a)$ shifts left by a units.
- $f(x)+a$ shifts up by a units.
- $f(x)-a$ shifts down by a units.

Reflections

- $-f(x)$: reflection along the x axis.
- $f(-x)$: reflection along the y axis.

Dilations (stretches)

- $f(ax)$: horizontal dilation
- $af(x)$: vertical dilation

Example 12

Find the centre and radius of $x^2 + y^2 - 6x - 4y + 4 = 0$, and then sketch it.

Example 13

From the graph of $y = \sqrt{x}$, deduce the graph of $y = -\sqrt{-x}$.

Definition 9

Dilate to stretch out and make larger

- Vertical dilation factor of 2: $(x, y) \mapsto (x, 2y)$

  Description: hold x axis constant, then stretch vertically.

- Vertical dilation factor of $\frac{1}{3}$: $(x, y) \mapsto \left(x, \frac{1}{3}y\right)$

  Description: hold x axis constant, then compress vertically.

- Horizontal dilation factor of 2: $(x, y) \mapsto (2x, y)$

  Description: hold y axis constant, then stretch horizontally.

- Horizontal dilation factor of $\frac{1}{5}$: $(x, y) \mapsto \left(\frac{1}{5}x, y\right)$

  Description: hold y axis constant, then compress horizontally.

1.5.1 Horizontal dilations (stretches from y axis)

Steps

To stretch a graph in the horizontal direction by a factor of a from the y axis (*horizontal dilation*), replace x with $\frac{x}{a}$, i.e. new function rule is

$$y = f\left(\frac{x}{a}\right)$$

1.5.2 Vertical dilations (stretches from x axis)

Steps

To stretch a graph in the vertical direction by a factor of a from the x axis (*vertical dilation*), replace y with $\frac{y}{a}$, i.e. new function rule is

$$y = af(x)$$

Example 14

[2025 NBHS Year 12 Assessment Task 2 Q10] (4 marks) The graph of $h(x)$ is transformed from the graph of $f(x) = 2x^3 + 1$, with transformations applied in the following order:

1. Reflection over the y-axis
2. Translation by one unit to the right
3. Horizontal dilation by a factor of four
4. Vertical translation by four units in an upwards direction

Find the equation of $h(x)$ in the form $h(x) = k(ax+b)^n + c$, where a, b, c, k & n are real numbers. **Answer:** $k=-2$, $a=\frac{1}{4}$, $b=-1$, $c=5$, $n=3$

Steps

Apply the listed transformations to $f(x) = 2x^3 + 1$ one-by-one, to each previously applied transformation:

1. Reflection over the y-axis

$$f_1(x) = f\left(-x\right)$$
$$= -2x^3 + 1$$

2. Translation by one unit to the right

$$f_2(x) = f_1\left(x - 1\right)$$
$$= -2(x-1)^3 + 1$$

3. Horizontal dilation by a factor of four

$$f_3(x) = f_2\left(\frac{x}{4}\right)$$
$$= -2\left(\frac{x}{4} - 1\right)^3 + 1$$

4. Vertical translation by four units in an upwards direction

$$f_4(x) = f_3\left(x\right) + 4$$
$$= -2\left(\frac{x}{4} - 1\right)^3 + 1 + 4$$
$$= -2\left(\frac{x}{4} - 1\right)^3 + 5$$

Example 15

[2022 CSSA Adv Trial Q26] (3 marks) Let $f(x) = x^2 - 2x$.

Sketch the graph of $y = 2f(1 - x) - 6$, showing the location of the vertex.

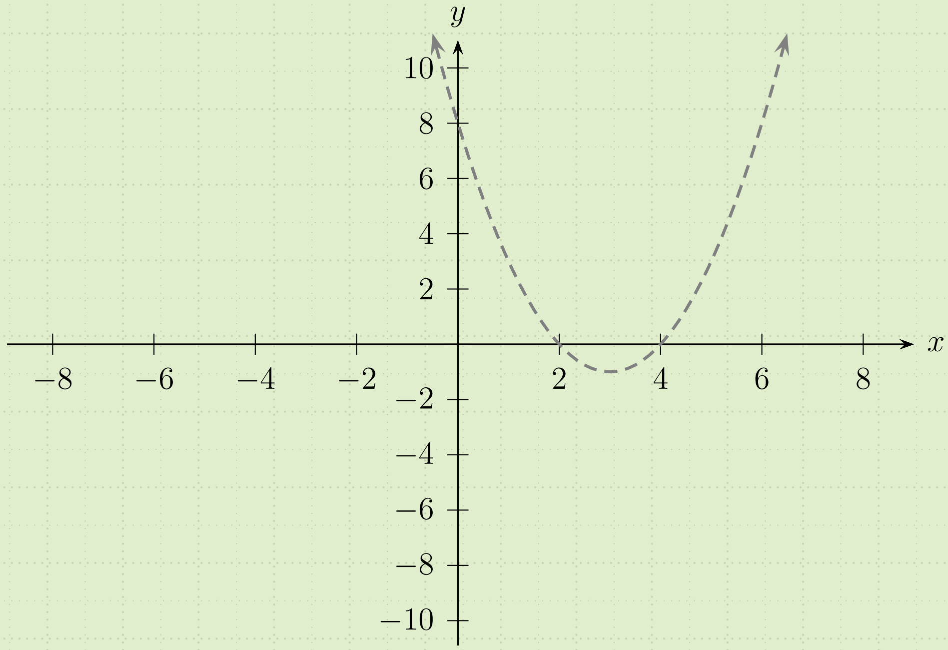
Example 16

[2024 NBSC Manly Campus Trial Q13] Let $f(x) = (x-3)^2 - 1$.

$f(x)$ is transformed into $g(x)$ such that $g(x) = \dfrac{1}{2}f(2-x)$.

(a) Complete a possible sequence of transformations that maps $f(x)$ to $g(x)$. **2**

(b) On the same set of axes below, sketch the graph of $y = g(x)$. **2**



Additional questions

Acknowledgement Portions taken from Evans et al. (2007, Ex 21C)

1. Let $f(x) = 2x + 3$. Sketch the graphs of:

(a) $y = f(x)$
(b) $y = f(x) + 4$
(c) $y = -f(x)$
(d) $y = -f(x) + 2$

2. Let $f(x) = 3x$. Sketch the graphs of:

(a) $y = f(x)$
(b) $y = f(x) + 4$
(c) $y = -f(x)$
(d) $y = -f(-x)$

3. Use transformations to sketch the graphs of each of the following functions and write down its maximal domain and range:

(a) $f(x) = x^2 + 5$
(b) $f(x) = (x - 5)^2$
(c) $f(x) = (x + 4)^2$
(d) $f(x) = x^2 - 3$
(e) $f(x) = 3^{-x}$
(f) $f(x) = 5^x + 1$
(g) $f(x) = 5^x - 4$
(h) $f(x) = 2 + \log_3 x$
(i) $f(x) = \log_3(x - 4)$

4. Sketch the graph of each function and write down its maximal domain and range:

(a) $f(x) = x^2 + 2$
(b) $f(x) = x^2 - 6x + 13$
(c) $f(x) = \sqrt{x}$
(d) $f(x) = \sqrt{2x} + 2$
(e) $f(x) = -\sqrt{x - 2}$
(f) $f(x) = 2 - \sqrt{x + 2}$

5. Let $f(x) = \sqrt{25 - x^2}$. On one set of axes, sketch the graphs of

(a) $y = f(x)$
(b) $y = f(x) + 2$
(c) $y = -f(x)$

6. Let $f(x) = x^3 - 3x^2 + 2x$. On one set of axes, sketch the graphs of

(a) $y = f(x)$
(b) $y = -f(x)$
(c) $y = -2f(x)$

7. Let $f(x) = \dfrac{1}{x}$. On one set of axes, sketch the graphs of

(a) $y = f(x)$
(b) $y = 2f(x)$
(c) $y = -f(x)$
(d) $y = -2f(x)$

Further exercises

(A) **Ex 4A** (Pender et al., 2019a)
- Q3-6, 8, 9, 11, 16, 17

(A) **Ex 4B**
- Q5

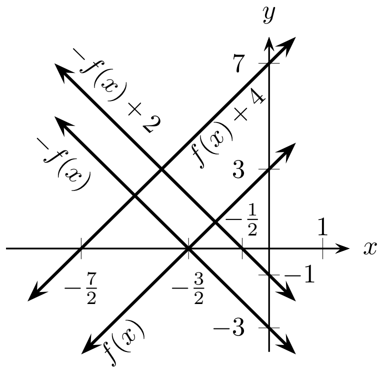
(X1) **Ex 4A** (Pender et al., 2019b)
- Q3, 5, 6, 11-14

(X1) **Ex 4B**
- Q5, 13

Answers

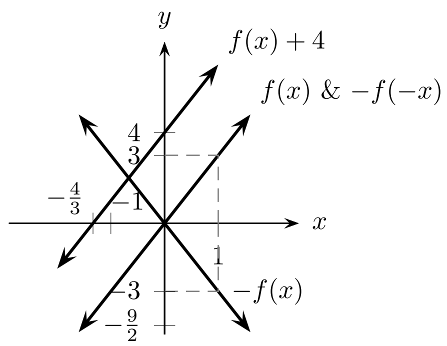
1. (a) $f(x) = 2x + 3$ (c) $-f(x) = -2x - 3$

(b) $f(x) + 4 = 2x + 7$ (d) $-f(x) + 2 = -2x - 1$



2. (a) $y = f(x) = 3x$ (c) $-f(x) = -3x$

(b) $f(x) + 4 = 3x + 4$ (d) $-f(-x) = 3x$



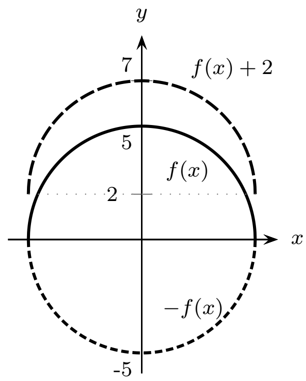
3. No sketch provided here. Instead, the direction & magnitude of shifting is shown with domain/range.

	Orig. fn	**Shift**	**Amt**	$D =$	$R =$
(a)	x^2	$\uparrow$	5	$x \in \mathbb{R}$	$y \geq 5$
(b)	x^2	$\rightarrow$	5	$x \in \mathbb{R}$	$y \geq 0$
(c)	x^2	$\leftarrow$	4	$x \in \mathbb{R}$	$y \geq 0$
(d)	x^2	$\downarrow$	3	$x \in \mathbb{R}$	$y \geq -3$
(e)	3^x	$\leftrightarrow$		$x \in \mathbb{R}$	$y \geq 0$
(f)	5^x	$\uparrow$	1	$x \in \mathbb{R}$	$y > 1$
(g)	5^x	$\downarrow$	4	$x \in \mathbb{R}$	$y > -4$
(h)	$\log_3 x$	$\uparrow$	2	$x > 0$	$y \in \mathbb{R}$
(i)	$\log_3 x$	$\rightarrow$	4	$x > 4$	$y \in \mathbb{R}$

4. No sketch provided here. Instead, the direction & magnitude of shifting + stretching is shown with domain/range.

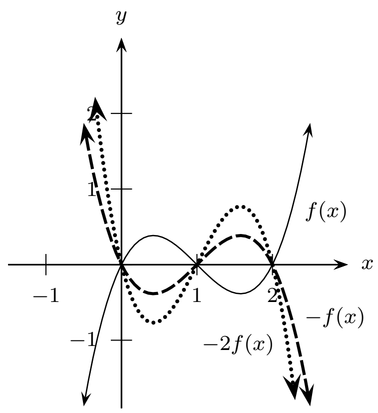
	Orig. fn	**Horiz shift**	**Vert shift**	**Reflect /stretch**	$D =$	$R =$
(a)	x^2	none	$\uparrow 2$	none	$\mathbb{R}$	$y \geq 2$
(b)	x^2	$\rightarrow 3$	$\uparrow 4$	none	$\mathbb{R}$	$y \geq 4$
(c)	$\sqrt{x}$	none	none	none	$x \geq 0$	$y \geq 0$
(d)	$\sqrt{x}$	none	$\uparrow 2$	$\updownarrow 2$	$x \geq 0$	$y \geq 2$
(e)	$\sqrt{x}$	$\rightarrow 2$	none	x axis	$x \geq 2$	$y \leq 0$
(f)	$\sqrt{x}$	$\leftarrow 2$	$\uparrow 2$	x axis	$x \geq -2$	$y \leq 2$

5.

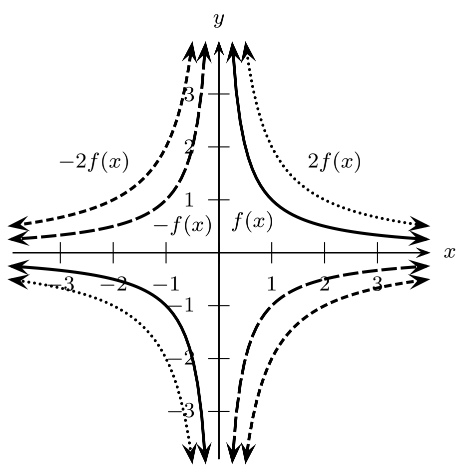


6. $y = x^3 - 3x^2 + 2x$. Factorising,

$$y = x(x-2)(x-1)$$



7. $f(x) = \frac{1}{x}$.



1.6 Odd and even functions

Definition 10

- An odd function has point symmetry about the origin. Algebraically,

$$f(-x) = -f(x)$$

- An even function has symmetry about the y axis. Algebraically,

$$f(x) = f(-x)$$

Important note

Observation:

- Odd polynomials only contain odd powers of x.
- Even polynomials only contain even powers of x.

Example 17

Test these functions to see whether they are odd, even or neither.

(a) $f(x) = x^4 - 3$. (b) $f(x) = x^3$. (c) $f(x) = x^2 - 2x$.

Example 18

[2023 Adv HSC Q9] Let $f(x)$ be any function with domain all real numbers.

Which of the following is an even function, regardless of the choice of $f(x)$?

(A) $2f(x)$ (B) $f\big(f(x)\big)$ (C) $\big(f(-x)\big)^2$ (D) $f(x)f(-x)$

Example 19

[2025 NBHS Year 11 Ext 1 Task 1 Q6] (2 marks) State whether $f(x) = \sqrt[3]{x}$ is odd, even or neither. Justify your answer with full mathematical reasoning.

Additional questions

1. State whether the following functions are *even*, *odd* or *neither*:

(a) $y = 2x^2$

(b) $y = 2x^3$

(c) $y = x^4 - x^2 + 1$

(d) $y = x - x^3 + 1$

(e) $y = \dfrac{x}{x^2 - 1}$

(f) $y = (x + 1)^2$

(g) $y = 2^x$

(h) $y = 2^{-x}$

(i) $y = 2^x + 2^{-x}$

(j) $y = 2^x - 2^{-x}$

2. (a) Sketch the graph of $y = x^3$, $0 \leq x \leq 2$. Show the coordinates of the end points.

(b) The above function is part of an even function $f(x)$, defined in the domain $-2 \leq x \leq 2$. Sketch $y = f(x)$.

(c) Write a piecewise description of the function $f(x)$.

3. (a) Sketch the graphs of $y = \sin x$ and $y = \cos x$ on separate number planes, each in the domain $-180° \leq x \leq 180°$.

(b) Hence decide if the sine and cosine functions are odd, even or neither.

(c) Remembering that $\sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}}$, write down the values of $\sin(-45°)$ and $\cos(-45°)$.

4. Show that if an odd function is defined for $x = 0$, then its graph must pass through the origin.

Further exercises

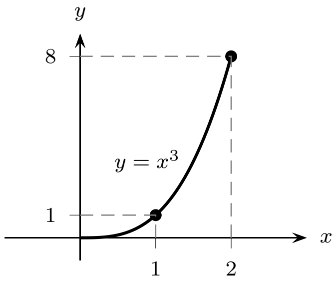
(A) **Ex 4C** (Pender et al., 2019a)
- Q1-7, 9-10, 12-13

(X1) **Ex 4C** (Pender et al., 2019b)
- Q1-7, 9-10, 12(a), 13-15

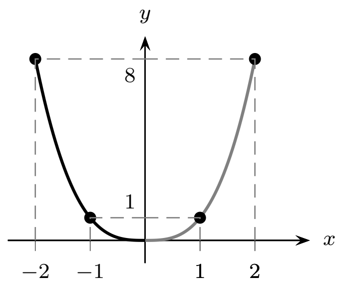
Answers

1. (a) even (b) odd (c) even (d) neither (e) odd (f) neither (g) neither (h) neither (i) even (j) odd

2. (a) $y = x^3$, $0 \le x \le 2$

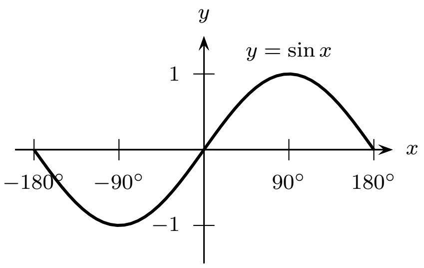


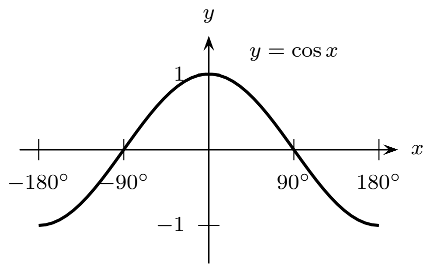
(b) Reflect $y = x^3$, $0 \le x \le 2$ about the y axis:



(c) $f(x) = \begin{cases} -x^3 & -2 \le x < 0 \\ x^3 & 0 \le x \le 2 \end{cases}$

3. (a) $y = \sin x$, $y = \cos x$, $-180° \le x \le 180°$





(b) $y = \sin x$: odd. $y = \cos x$: even

(c) $\sin(-45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$.
$\cos(-45°) = \cos 45° = \frac{1}{\sqrt{2}}$.

Section 2

Various families of functions

Learning Goal(s)

Knowledge
All basic curves

Skills
Sketching these basic curves with transformations

Understanding
Basic applications of these curves

By the end of this section am I able to:

2.3 Quadratic and cubic functions

- Identify the x-intercepts of a parabola whose quadratic function is expressed in factored form
- Use the discriminant to determine the number of x-intercepts on a parabola and justify its position in relation to the x-axis
- Show by completing the square on the general quadratic $y = ax^2+bx+c$ that the axis of symmetry is $x = -\frac{b}{2a}$ where a, b and c are constants
- Identify the axis of symmetry and vertex of a parabola by completing the square on its quadratic function
- Choose and apply appropriate techniques to graph a parabola of the form $y = ax^2 + bx + c$ by identifying its x-intercepts if they exist, its y-intercept, its axis of symmetry using $x = -\frac{b}{2a}$ and its vertex
- Find the equation of a parabola given sufficient graphical features
- Use the fact that two quadratic functions are equal for all values of x if and only if the corresponding coefficients are equal to solve related problems
- Model and solve practical problems involving quadratic functions and justify conclusions in the context of the problem
- Recognise that solving $f(x) = k$ for some constant k corresponds to finding the x-coordinate(s) of the intersection of the graphs $y = f(x)$ and $y = k$
- Solve problems by finding the solution to simultaneous equations involving a linear and a quadratic function, or two quadratic functions, both algebraically and graphically
- Solve quadratic inequalities
- Recognise and graph cubic functions of the form $f(x) = kx^3$ and $f(x) = k(x-a)(x-b)(x-c)$, where a, b, c and k are constants and $k \neq 0$

Important note

Some of these families of curves have already been covered in the previous sections. Refer to previous textbook exercises for further practice.

2.1 Linear functions

Important note

No new theory is in this section. However, expect slightly more difficult problems within textbook exercises.

Definition 11

The **linear function** takes on the **gradient-intercept form** of the equation of a line:

$$y = mx + c$$

where

- m – gradient
- c – y intercept

Laws/Results

The gradient-intercept form can be rearranged into the **general form** of the equation of a line:

$$ax + by + c = 0$$

Laws/Results

The **point-gradient formula** to find the equation of a straight line through (x_1, y_1) with a given gradient m:

$$\frac{y - y_1}{x - x_1} = m$$

2.2 Quadratic functions

2.2.1 Geometry of a parabola

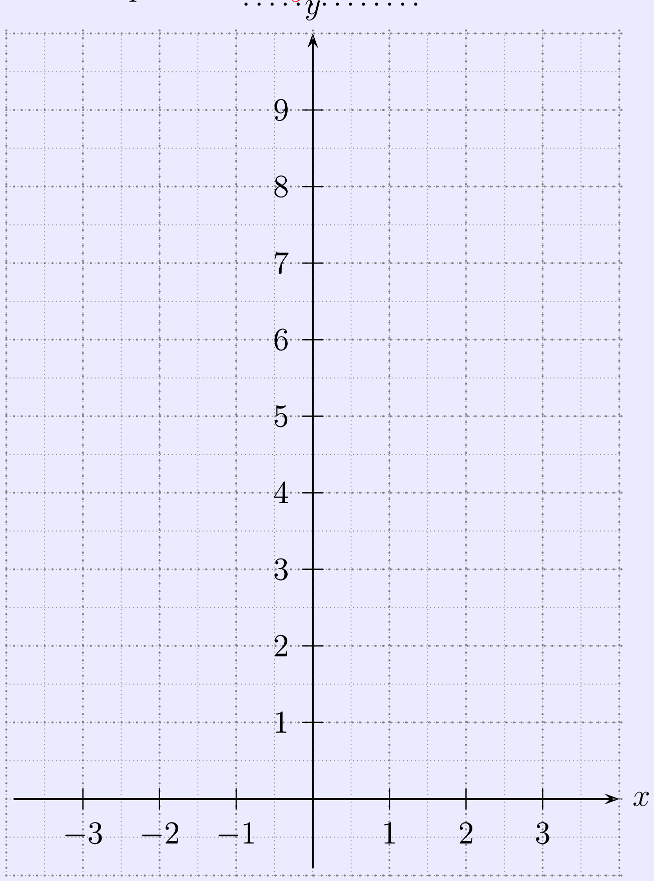
Definition 12

A *parabola* is the geometric representation (graph) of the *quadratic polynomial.*

Basic parabola

Definition 13

The basic parabola has equation ....$y = x^2$.... .



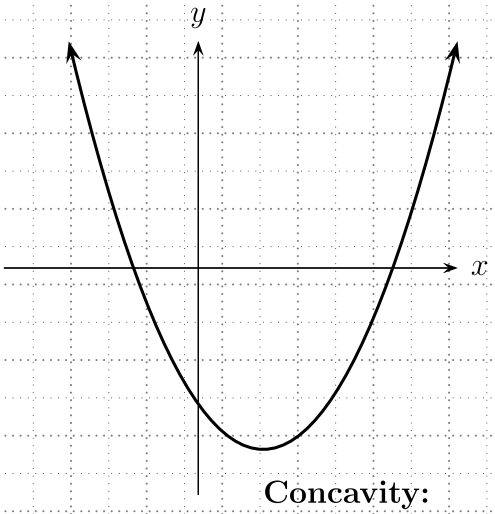
- The **vertex** (....minimum.... value) is at ....$(0, 0)$....
- The **axis of symmetry** is at ....$x = 0$....

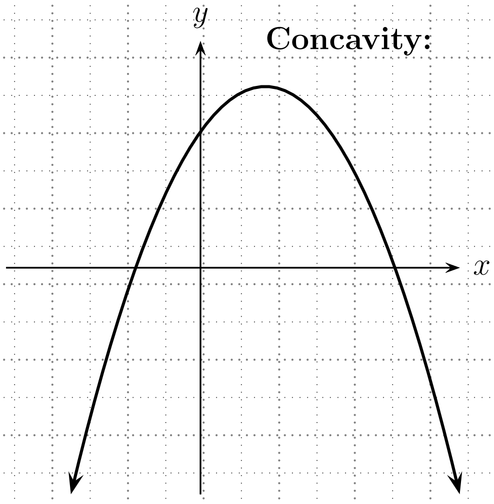
GeoGebra

quadratic.ggb
www.desmos.com

General parabolas

x intercept	**y intercept**	**minimum value (y)**	
maximum value (y)	**turning point**	**vertex**	**axis**





General form and roots

Definition 14

The *general form* of a quadratic polynomial:

$$y = ax^2 + bx + c$$

where c is the y intercept of the parabola.

The highest power (degree) of a quadratic: 2

Definition 15

The *roots* or *zeros* of the quadratic polynomial are the x intercepts, if any.

Graph representative of the general form:

2.2.2 Factorisation and the graph

Definition 16

The *factored form* of quadratic polynomial:

$$y = a(x-\alpha)(x-\beta)$$

where

- α and β are the *roots* of the quadratic polynomial
- a is the vertical dilation factor.

Graph representative of the factored form:

Example 20

Find the roots of $y = x^2 + 5x + 6$.

Important note

Factorisation techniques from Stage 4-5 (Years 8-10) are required!

Example 21

Sketch the following graphs:

(a) $y = x^2 - 6x + 8$

(b) $y = -2x^2 + 9x - 7$

Example 22

[Ex 8A Q7] Write down the general form of a monic quadratic for which one of the zeroes of $x = 1$. Then find the equation of such a quadratic in which

(a) the curve passes through the origin,

(b) there are no other zeroes,

(c) the axis of symmetry is $x = -7$,

(d) the curve passes through $(3, 9)$.

Laws/Results

The ..... vertical .... dilation .... factor ..... needs to be considered when two roots are given and the parabola's equation needs to be formed.

Example 23

Write down the equation of a parabola with zeroes -2 and 4, as well as

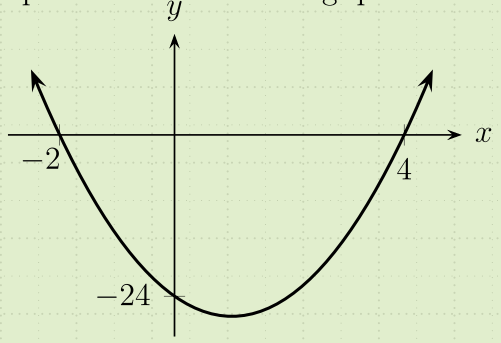
(a) y intercept of 6

(b) vertex at $(1, 21)$

Answer: (a) $a = -\frac{3}{4}$ (b) $a = -\frac{7}{3}$

Example 24

[Ex 8A Q12] Find the equation of the following quadratic:



2.2.3 Quadratic inequalities

Example 25

Solve $x^2 - 6x + 5 \geq 0$.

Important note

⚠ Draw picture!

Example 26

[2023 Adv HSC Q19]

(a) Sketch the graphs of the functions $f(x) = x - 1$ and $g(x) = (1 - x)(3 + x)$, showing the x intercepts. **2**

(b) Hence, or otherwise, solve the inequality $x - 1 < (1 - x)(3 + x)$ **2**

Timed exam practice 1 (Allow approximately 13 minutes)

[**2025 NBHS Year 11 Ext 1 Task 1 Q6**]

i. Solve for x: $(6x+1)(x+1) \geq 0$. **3**

ii. Given $f(x) = \sqrt{3x^2+x-2}$ and $g(x) = 2x+1$.

(A) Find $f\big(g(x)\big)$ in the form $\sqrt{ax^2+bx+c}$, where a, b and c are integers. **2**

(B) Hence find the the domain of $f\big(g(x)\big)$. **3**

Answer: i. $x \leq -1$ or $x \geq -\frac{1}{6}$ ii. A. $f\big(g(x)\big) = \sqrt{12x^2+14x+2}$ B. $x \leq -1$ or $x \geq -\frac{1}{6}$

Marking criteria

i. (3 marks)

- ✓ [1] For sketching a graph that resembles a parabola.
- ✓ [2] For the parabola $y = (6x+1)(x+1)$ being drawn correctly, showing correct x intercepts, or equivalent merit.
- ✓ [3] For fully correct solution.

ii. (A) (2 marks)

- ✓ [1] For substantial progress to simplifying $f(2x+1)$.
- ✓ [2] For fully correct solution.

(B) (3 marks)

- ✓ [1] For correctly creating the inequality to find the domain of $f\big(g(x)\big)$
- ✓ [2] For substantial progress to solving the inequality created.
- ✓ [3] For fully correct solution.

Domain of $f\big(g(x)\big)$: function inside the square root must be positive or zero.

Further exercises

Ⓐ **Ex 3D** (Pender et al., 2019a)

- All questions

Ⓧ1 **Ex 3D** (Pender et al., 2019b)

- Q5-16
- Q17, 18

2.2.4 Completing the square and the graph

Definition 17

The *vertex form* of quadratic polynomial:

$$y = \dots a(x-h)^2 + k \dots$$

where

- (h, k) is the ... vertex ... of the parabola
- a is the ... vertical ... stretch ... factor ... .

Graph representative of the vertex form:

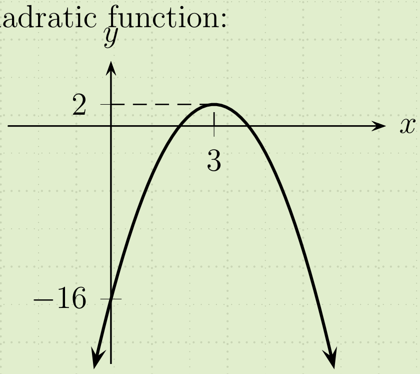
Example 27

Sketch the following parabolas:

(a) $y = x^2 + x + 1$ (b) $y = -3x^2 - 4x + 2$

Example 28

Find the equation of the quadratic function:



Further exercises

Ⓐ **Ex 3E** (Pender et al., 2019a)

- All questions

Ⓧ1 **Ex 3E** (Pender et al., 2019b)

- Q9(d) - (f)
- Q10-18
- Q19, 20

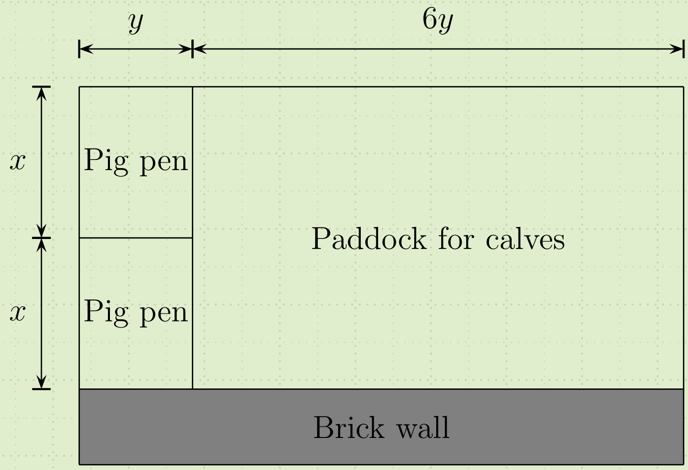
2.2.5 Optimisation with quadratics

Steps

1. Show the results required.
2. With the quadratic obtained, complete the square to use vertex form, or find the location of the vertex using $x = -\frac{b}{2a}$.
3. Answer the question

Example 29

[**1988 2U HSC Q9**] Farmer Brown wishes to construct three rectangular enclosures, as shown, in which to put pigs and calves. The paddock for the calves is to be six times as long and twice as wide as a pig pen. One pig pen and the calves' paddock have an existing brick wall as a boundary fence as shown. All other fences are to be constructed from 56 metres of wire mesh.



i. Let x metres be the width of the pig pen and y metres be its length. Show that **2**

$$y = 7 - \frac{3}{4}x$$

ii. Hence show that the total area A square metres contained in the three enclosures is given by **2**

$$A = 14x\left(7 - \frac{3}{4}x\right)$$

iii. Show that A is a maximum when half the wire fencing has been placed parallel to the brick wall. **3**

Important note

Do not skip steps if you are required to **show** particular results!

Marking criteria

i. ✓ [1] Relates x and y to the total amount of fencing available, or equivalent merit.
✓ [2] Correct solution.

ii. ✓ [1] Relates x and y to the total area to be enclosed, or equivalent merit.
✓ [2] Correct solution.

iii. ✓ [1] Finds the correct value of x for where the maximum area would occur, or equivalent merit.
✓ [2] Finds the value of y and relates it to half of the wire fencing available, or equivalent merit.
✓ [3] Correct solution.

2.2.6 The quadratic formula and the graph

Laws/Results

The quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

allows the

- roots of the equation $ax^2 + bx + c = 0$, as well as the
- x intercepts of the function $y = ax^2 + bx + c$

to be found, where they exist.

Steps

Derivation of the quadratic formula to solve $ax^2 + bx + c = 0$

1. Rewrite as monic quadratic:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

2. Complete the square on x:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

3. Simplify and rewrite with x as subject:

Further exercises

Ⓐ **Ex 3F** (Pender et al., 2019a)
- Q1-5, 9-10

(X1) **Ex 3F** (Pender et al., 2019b)
- Q6-13
- Q14

2.2.7 Quadratic identities

Definition 18

Identities are true for *all* values of x, i.e.

- $f(x) = g(x)$ may be true only at their ....intersection........ ....points.... .
- $f(x) \equiv g(x)$ is stating that $f(x)$ *is* $g(x)$.

Theorem 2

Two quadratics are *identical* if and only if all of their corresponding coefficients are equal, i.e. $f(x) = ax^2 + bx + c$ is identical to $g(x) = rx^2 + sx + t$ if and only if

$$a = r \qquad b = s \qquad c = t$$

Counterexample:

Example 30

[2013 Independent Year 11 2U Yearly Q14] (3 marks) Find the values of a, b and c if

$$2x^2 + 3x + 1 = ax(x+2) + b(x+2) + c$$

Answer: $a = 2$, $b = -2$, $c = 3$

Important note

Either

- Gather powers of x and compare coefficients, or
- Find "convenient" values of x to substitute!

Important note

Essentially, most questions are merely "cosmetic surgery" on a quadratic.

Example 31

Find constants a and b such that

$$x^2 + 10x + 10 \equiv a(x+2)^2 + b(x+1)$$

Example 32

(a) Express $x^4 - 6x^3 + 7x^2 + 6x - 8 = 0$ in the form $\left(x^2 + ax\right)^2 + b\left(x^2 + ax\right) + c$.

(b) Hence solve the equation $x^4 - 6x^3 + 7x^2 + 6x - 8 = 0$.

Exercise 8I

1. Show that the following quadratic identity holds when $x = a$, $x = b$ and $x = 0$, where a and b are distinct and nonzero. Explain why it follows that it is true for all values of x.

$$x(x-b) + x(x-a) \equiv x(2x-a-b)$$

2. Show that $x = 0$, $x = a$ and $x = b$ are all solutions of the quadratic equation

$$\frac{x(x-a)}{b-a} + \frac{x(x-b)}{a-b} = x,$$

where a and b are distinct and nonzero. For what values of x does it now follow that this equation is true?

3. Similarly prove the following identities in x, where a, b and c are distinct:

(a) $\dfrac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \dfrac{b^2(x-c)(x-a)}{(b-c)(b-a)} + \dfrac{c^2(x-a)(x-b)}{(c-a)(c-b)} \equiv x^2$

(b) $(b-c)(x-a) + (c-a)(x-b) + (a-b)(x-c) \equiv 0$

NOTE: The last identity is only linear, and so only two values of x need be tested.

4. (a) If $x^2 + x + 1 = a(x-1)^2 + b(x-1) + c$ for all values of x, form three equations by substituting $x = 0$ and $x = 1$, and equating coefficients of x^2. Hence find a, b and c.

(b) Find a, b and c if $n^2 - n \equiv a(n-4)^2 + b(n-4) + c$. [HINT: Substitute $n = 4$, substitute $n = 0$, and equate coefficients of n^2.]

5. (a) Express $2x^2 + 3x - 6$ in the form $a(x+1)^2 + b(x+1) + c$.

(b) Find a, b and c if $2x^2 + 4x + 5 \equiv a(x-3)^2 + b(x-3) + c$.

(c) Express $2x^2 - 5x + 3$ in the form $a(x-2)^2 + b(x-2) + c$.

6. (a) Express x^2 in the form $a(x+1)^2 + b(x+1) + c$.

(b) Express n^2 as a quadratic in $(n-4)$.

(c) Express x^2 as a quadratic in $(x+2)$.

DEVELOPMENT

7. Prove the following identities, where p, q and r are distinct:

(a) $(p+q+x)(pq+px+qx) - pqx \equiv (p+q)(p+x)(q+x)$

(b) $\dfrac{p(x-q)(x-r)}{(p-q)(p-r)} + \dfrac{q(x-p)(x-r)}{(q-r)(q-p)} + \dfrac{r(x-p)(x-q)}{(r-p)(r-q)} \equiv x$

(c) $\dfrac{(p-x)(q-x)}{(p-r)(q-r)} + \dfrac{(q-x)(r-x)}{(q-p)(r-p)} + \dfrac{(r-x)(p-x)}{(r-q)(p-q)} \equiv 1$

8. (a) If $2x^2 - x - 1 = a(x-b)(x-c)$ for all real values of x, find a, b and c.

(b) Express m^2 in the form $a(m-1)^2 + b(m-2)^2 + c(m-3)^2$.

9. (a) Express $2x - 5$ in the form $A(2x+1) + B(x+1)$.

(b) Express $\dfrac{2x+3}{(x+1)(x+2)}$ in the form $\dfrac{A}{x+1} + \dfrac{B}{x+2}$. [HINT: You will need to multiply both sides of $\dfrac{2x+3}{(x+1)(x+2)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x+2}$ by $(x+1)(x+2)$.]

(c) Express $\dfrac{x+1}{x(x+2)}$ in the form $\dfrac{A}{x} + \dfrac{B}{x+2}$.

EXTENSION

12. (a) Express $\dfrac{x^2}{(x-m)(x-n)}$ in the form $A + \dfrac{B}{x-m} + \dfrac{C}{x-n}$. Hence or otherwise, find the value of $\dfrac{m^2}{(m-n)(m-r)} + \dfrac{n^2}{(n-m)(n-r)} + \dfrac{r^2}{(r-m)(r-n)}$, where m, n and r are distinct.

(b) Similarly, given that the theorems proven in this section may be extended to cubics, find the value of $\dfrac{m^3}{(m-n)(m-r)(m-p)} + \dfrac{n^3}{(n-m)(n-r)(n-p)}$
$+ \dfrac{r^3}{(r-m)(r-n)(r-p)} + \dfrac{p^3}{(p-m)(p-n)(p-r)}$, where m, n, r and p are distinct.

Exercise 8I (Page 313)

2 It is true for all values of x, and hence is an identity.

4(a) $a = 1,\ b = 3,\ c = 3$ **(b)** $a = 1,\ b = 7,\ c = 12$

5(a) $2(x+1)^2 - (x+1) - 7$

(b) $a = 2,\ b = 16$ and $c = 35$

(c) $2(x-2)^2 + 3(x-2) + 1$

6(a) $(x+1)^2 - 2(x+1) + 1$ **(b)** $(n-4)^2 + 8(n-4) + 16$

(c) $(x+2)^2 - 4(x+2) + 4$

8(a) $a = 2,\ b = 1,\ c = -\frac{1}{2}$ or $a = 2,\ b = -\frac{1}{2},\ c = 1$

(b) $3(m-1)^2 - 3(m-2)^2 + (m-3)^2$

9(a) $7(2x+1) - 12(x+1)$ **(b)** $\dfrac{1}{x+1} + \dfrac{1}{x+2}$

(c) $\dfrac{1}{2x} + \dfrac{1}{2(x+2)}$

12(a) 1 **(b)** 1

Further exercises

Ⓐ **Ex 10I** (Pender et al., 2009)

- All questions

ⓍⒶ **Ex 8I** (Pender et al., 1999)

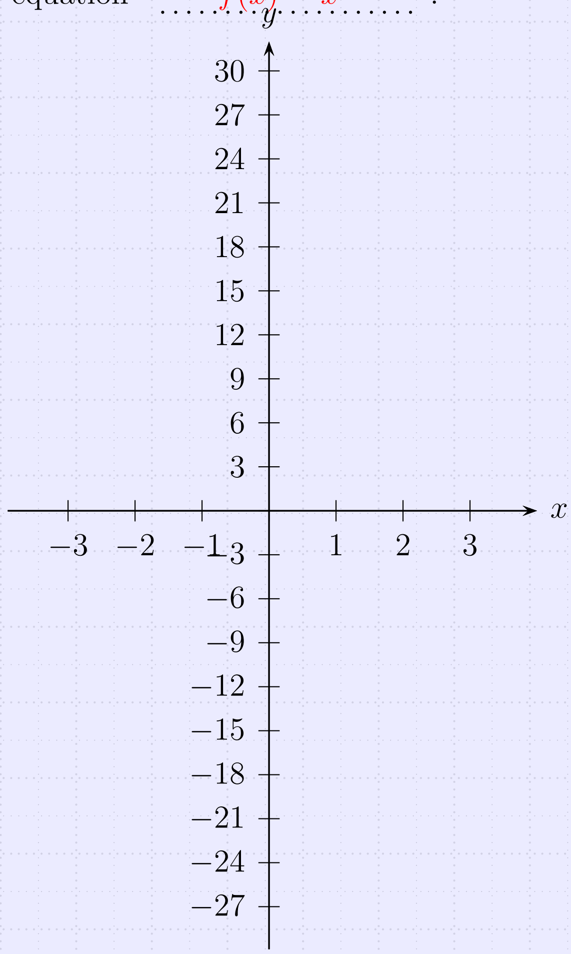
- Q2-4, 6, 7(c), 8
- Q9, 12(a)

2.3 Cubics

2.3.1 Basic cubics

Definition 19

The basic cubic has equation $f(x) = x^3$.



- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $R = \{y : y \in \mathbb{R}\}$
- Generalised equation: $y = a(x-h)^3 + k$, a is the vertical dilation factor.
- The horizontal point of inflexion is at $(0, 0)$
- The cubic function in its most basic form is odd

Example 33

Sketch $y = x^3 - 3$, showing all important features.

Example 34

Sketch $y = (x-2)^3 + 3$, showing all important features.

2.3.2 Cubics (factored)

Definition 20

Other cubics may be in **factored form** with equation

$$f(x) = k(x-a)(x-b)(x-c)$$

- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $R = \{y : y \in \mathbb{R}\}$
- a is the vertical dilation factor.
- The zeroes of the cubic are at $x = a$, $x = b$ and $x = c$

Theorem 3

Behaviour of single, double and triple roots

Fully factored term	**Algebra**	**Behaviour near zero**	**Multiplicity**
Linear			
Quadratic	$(x-2)^2$		
Cubic		Cubic	

Example 35

Sketch the following cubics:

(a) $y = (x+1)(x-2)(x-3)$

(b) $y = x(x-2)^2$

(c) $y = (x-2)^3(x+1)$

Important note

All of the roots, plus the y intercept should be shown clearly.

Example 36

[2020 Sydney Girls HS Adv Trial Q10] The graph of $y = (x-2)^2(3+x)$ is dilated vertically by a factor of $\frac{1}{2}$ and horizontally by a factor of 3. It is then reflected across the y axis.

Which of the following is the new equation?

(A) $y = -2(3x-2)^2(3+3x)$

(B) $y = \frac{1}{2}(3x-2)^2(3+3x)$

(C) $y = -\frac{1}{2}\left(\frac{x}{3}-2\right)^2\left(3+\frac{x}{3}\right)$

(D) $y = \frac{1}{2}\left(\frac{x}{3}+2\right)^2\left(3-\frac{x}{3}\right)$

Additional questions

1. Sketch the graphs for

(a) $y = (x-1)(x-3)$

(b) $y = x(x+2)(x-3)$

(c) $y = -x(x-1)(x-2)$

(d) $y = (x-2)^2$

(e) $y = (x+1)^3$

(f) $y = x(x-3)^2$

(g) $y = (x+1)^2(x-1)$

2. Sketch the graphs for

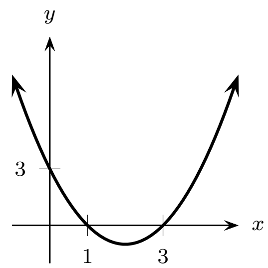
(a) $y = x^3 + 3x^2 + 2x$

(b) $y = x^3 - 6x^2 + 9x$

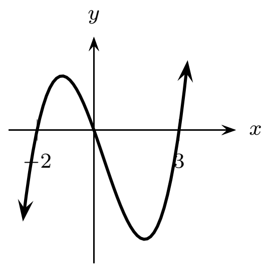
(c) $y = x^3 + 2x^2 - x - 2$

Answers

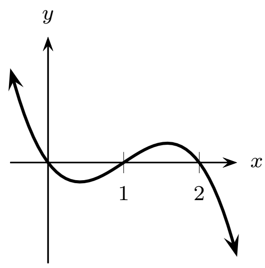
1. (a) $y = (x-1)(x-3)$



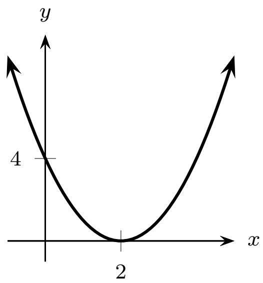
(b) $y = x(x+2)(x-3)$



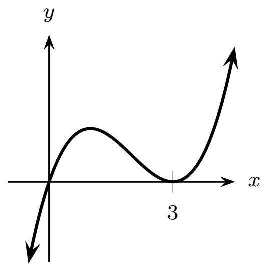
(c) $y = -x(x-1)(x-2)$



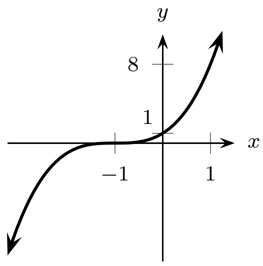
(d) $y = (x-2)^2$



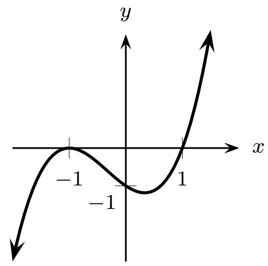
(f) $y = x(x-3)^2$



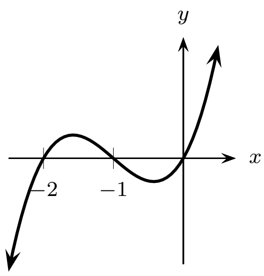
(e) $y = (x+1)^3$



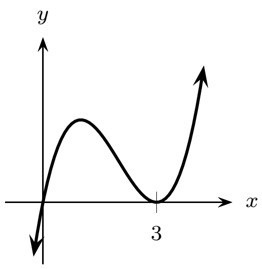
(g) $y = (x+1)^2(x-1)$



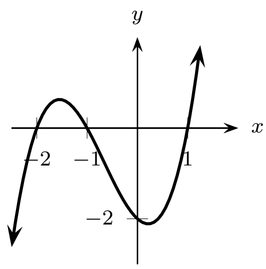
2. (a) $y = x^3 + 3x^2 + 2x$



(b) $y = x^3 - 6x^2 + 9x$



(c) $y = x^3 + 2x^2 - x - 2$



Further exercises (Legacy Textbooks)

Source McSeveney et al. (1986, Ex 15:07). Check solutions via GeoGebra or Desmos.

1. The following polynomials are given in factored form. Determine their zeros and sketch their graphs indicating clearly where they cut the x axis.

(a) $y = (x+1)(x-3)$

(b) $y = 2x(x-5)$

(c) $y = x(x+1)(x-1)$

(d) $y = (x-2)(x-1)(x+1)$

(e) $y = (x+4)(x+1)(x-2)$

(f) $y = x(x-3)(x+5)$

(g) $y = -x(x+2)(x-1)$

(h) $y = (3-x)(x+1)(x+2)$

(i) $y = (x-2)(x+1)(1-x)$

(j) $y = (2-x)(x+1)(x+4)$

2. Each polynomial function has a repeated root. Sketch the graph of each neatly.

(a) $y = (x-1)^2$

(b) $y = -(x+2)^2$

(c) $y = x(x+2)^2$

(d) $y = (x+1)(x-2)^2$

(e) $y = (x+3)^2(x-1)$

(f) $y = -x(x-1)^2$

(g) $y = x^2(5-x)$

(h) $y = (x+3)^3$

(i) $y = -(x+3)(x-5)^2$

(j) $y = (5-x)^3$

2.4 Circles

Learning Goal(s)

Knowledge
All basic curves

Skills
Sketching these basic curves with transformations

Understanding
Basic applications of these curves

By the end of this section am I able to:

2.7 Circles and semicircles

- Derive the equation of a circle of radius r with centre at the origin by considering Pythagoras' theorem
- Graph circles of the form $x^2 + y^2 = r^2$ from their equations
- Determine the equation of a circle of the form $x^2 + y^2 = r^2$ given its graph
- Identify and graph the semicircles

$$y = \sqrt{r^2 - x^2} \qquad y = -\sqrt{r^2 - x^2} \qquad x = \sqrt{r^2 - y^2} \qquad x = -\sqrt{r^2 - y^2}$$

Definition 21

The general equation of a circle (relation) with radius r, centred at $C(h,k)$ is

$$(x-h)^2 + (y-k)^2 = r^2$$

- Domain: $D = \{x : x \in [h-r, h+r]\}$
- Range: $R = \{y : x \in [k-r, k+r]\}$

Derivation of general equation

Steps

1. Let

- $P(x,y)$ be the point that which is equidistant from a fixed point.
- (h,k) be the fixed point
- r be the fixed distance.

2. Apply the distance formula:

....

....

....

....

Sketch

Definition 22

The general equation of a semicircle (function) with radius r, centred at $C(0,0)$ is

$$y = \sqrt{r^2 - x^2}$$

Sketch

Example 37

[2017 NBHS Year 11 2U Task 1 Q5] (2 marks) By drawing a quick sketch or otherwise, state the domain and range of

$$y = -\sqrt{13 - x^2}$$

Timed exam practice 2 (Allow approximately 5 minutes)

[2020 Adv HSC Q24] (3 marks) The circle $x^2 - 6x + y^2 + 4y - 3 = 0$ is reflected in the x axis.

Sketch the reflected circle, showing the coordinates of the centre and the radius.

Marking criteria

✓ [1] Completes the square for the equation, or equivalent merit

✓ [2] Finds the centre and the radius of the original circle or of the reflected circle, or equivalent merit

✓ [3] Provides the correct solution

Additional questions

Source McSeveney et al. (1986, Ex 4:09)

1. Use the formula to find equation of the following circles:

(a) centre $(1, 1)$ radius 7 units
(b) centre $(5, 0)$ radius 2 units
(c) centre $(-3, -5)$ radius 4 units
(d) centre $(2, -5)$ radius 1 units
(e) centre $(0, 2)$ radius $\frac{1}{2}$ units
(f) centre $(0, 0)$ radius 3 units

2. Find the centre and radius of each of the following circles, then sketch each of the following on separate number planes:

(a) $(x-2)^2 + (y-3)^2 = 64$
(b) $(x+4)^2 + (y-1)^2 = 4$
(c) $(x+3)^2 + (y+3)^2 = 9$
(d) $(x-6)^2 + (y-5)^2 = 100$
(e) $x^2 + (y+5)^2 = 16$
(f) $(x-3)^2 + y^2 = 1$
(g) $x^2 + y^2 = 81$
(h) $x^2 + y^2 = 49$
(i) $x^2 + y^2 = 11$
(j) $(x-7)^2 + (y-8)^2 = 2$

3. Find the centre and radius of each of the following circles.

(a) $x^2 - 10x + y^2 + 8y + 32 = 0$
(b) $x^2 + y^2 + 8x - 14y = 35$
(c) $x^2 + y^2 - 18x - 20y + 60 = 0$
(d) $x^2 + y^2 - 9x + \frac{53}{4} = 0$

Answers

1. (a) $(x-1)^2+(y-1)^2 = 49$ (b) $(x-5)^2+(y-0)^2 = 4$ (c) $(x+3)^2+(y+5)^2 = 16$ (d) $(x-2)^2+(y+5)^2 = 1$ (e) $x^2+(y-2)^2 = \frac{1}{4}$ (f) $x^2 + (y-0)^2 = 9$ **2.** (a) centre $(2, 3)$ radius 8 units (b) centre $(-4, 1)$ radius 2 units (c) centre $(-3, -3)$ radius 3 units (d) centre $(6, 5)$ radius 10 units (e) centre $(0, -5)$ radius 4 units (f) centre $(3, 0)$ radius 1 units (g) centre $(0, 0)$ radius 9 units (h) centre $(0, 0)$ radius 7 units (i) centre $(0, 0)$ radius $\sqrt{11}$ units (j) centre $(7, 8)$ radius $\sqrt{2}$ units **3.** (a) centre $(5, -4)$ radius 3 units (b) centre $(-4, 7)$ radius 10 units (c) centre $(9, 10)$ radius 11 units (d) centre $\left(\frac{9}{2}, 0\right)$ radius $\sqrt{7}$ units

2.5 Exponential functions

Learning Goal(s)

Knowledge
All basic curves

Skills
Sketching these basic curves with transformations

Understanding
Basic applications of these curves

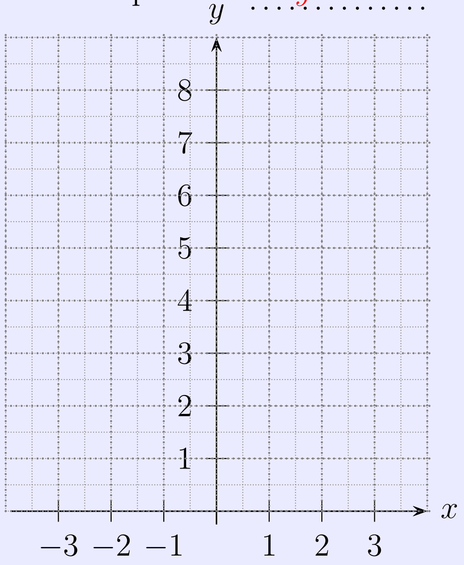
By the end of this section am I able to:

2.4 Exponential functions

- Graph the exponential functions $y = k(a^x)$ and $y = k(a^{-x})$ for constants a and k where $a > 0$, $a \neq 1$ and $k \neq 0$, and identify its asymptote, y-intercept, domain and range
- Describe the behaviour of $y = k(a^x)$ and $y = k(a^{-x})$ as $x \to \infty$ and $x \to -\infty$

Definition 23

A basic exponential curve has equation $y = a^x$



- Condition on a: $a > 0$
- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $D = \{x : y \in (0, \infty)\}$
- When $x = 0$, $y = 1$
- When $x = 1$, $y = a$
- As
  - $x \to \infty$, $y \to \infty$
  $$\lim_{x \to \infty} a^x = \infty$$
  - $x \to -\infty$, $y \to 0^+$
  $$\lim_{x \to -\infty} a^x = 0^+$$
- Generalised equation: $y = ka^{x-b} + c$.

Important note

When sketching, always show two points on the curve!

Example 38

Sketch the graph of $y = \left(\frac{1}{2}\right)^x$, showing all important features.

Example 39

Sketch the graph of $y = -2^{x+1}$, showing all important features.

2.6 Hyperbolas (upright rectangular)

Learning Goal(s)

Knowledge
All basic curves

Skills
Sketching these basic curves with transformations

Understanding
Basic applications of these curves

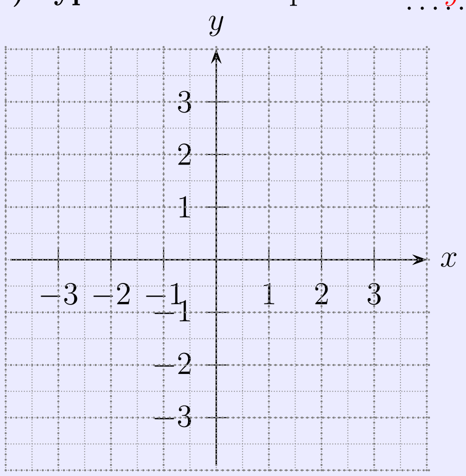
By the end of this section am I able to:

2.5 Reciprocal functions

- Graph functions of the form $f(x) = \frac{k}{x}$, where k is a constant and $k \neq 0$, and identify their hyperbolic shape and their asymptotes
- Describe the behaviour of $f(x) = \frac{k}{x}$ as $x \to \infty$ and $x \to -\infty$

Definition 24

The (upright rectangular) hyperbola has equation $y = \dfrac{1}{x}$.



- Domain: $D = \{x : x \in \mathbb{R} \setminus \{0\}\}$
- Range: $R = \{y : x \in \mathbb{R} \setminus \{0\}\}$
- Behaviour at extremities: as $x \to \pm\infty$, $y \to 0$, i.e.

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0$$

- Behaviour near asymptotes: as $x \to 0^+$, $y \to \infty$, i.e.

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

As $x \to 0^-$, $y \to -\infty$, i.e.

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

This function is discontinuous at $x = 0$.

- Generalised equation: $y = \frac{a}{x-h} + k$

Important note

Other hyperbolae exist, but they may not be upright or rectangular. General equation of a rectangular hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a is the length of the semi-major axis, and $a, b > 0$.

Timed exam practice 3 (Allow approximately 8 minutes)

[**2025 NBHS Year 11 Adv Task 1 Q7**]

(a) Carefully state the transformations that are required to transform $y = \dfrac{1}{x}$ to become $y = -\dfrac{1}{x-2} + 3$. **3**

(b) Sketch the graph of $y = -\dfrac{1}{x-2} + 3$, showing any intercepts with the axes and asymptotes. **2**

Marking criteria

(a) ✓ [1] Provides one of the three correct transformations using correct mathematical language, but not necessarily in the correct order.

✓ [2] Provides two of the three correct transformations using correct mathematical language, but not necessarily in the correct order.

✓ [3] Provides the correct solution, i.e. all three transformations using correct mathematical language, in the correct order.

(b) ✓ [1] For drawing a graph that has the shape, with the horizontal and vertical asymptote that resembles $y = \dfrac{1}{x}$.

✓ [2] For fully correct graph, with all intercepts with the x, y axes, as well as correct equations of asymptotes.

Example 40

Sketch $y = \dfrac{-2}{x+1} + 3$, showing all important features.

Example 41

[**2020 CSSA Adv Trial Q10**] The graph of $y = \dfrac{3}{x+1}$ is translated 4 units right and dilated vertically by a factor of $\dfrac{1}{2}$. Which of the following gives the equation of the new function?

(A) $\dfrac{y}{2} = \dfrac{3}{x-3}$ (B) $2y = \dfrac{3}{x-4}$ (C) $2y = \dfrac{3}{x-3}$ (D) $\dfrac{y}{2} = \dfrac{3}{x-4}$

Section 3

Variation

Learning Goal(s)

Knowledge
Direct and inverse proportion to x^n

Skills
Finding the constants of proportionality

Understanding
Differences between direct and inverse variation

By the end of this section am I able to:

2.6 Direct and inverse variation

- Develop models of the form $y = kx^n$, where k is a non-zero constant, from descriptions of situations in which one quantity varies directly with another
- Develop the model $y = \frac{k}{x^n}$, where k is a non-zero constant, from descriptions of situations in which one quantity varies inversely with another
- Evaluate k in the equations $y = kx^n$ and $y = \frac{k}{x^n}$, given one pair of values for the variables, and use the resulting formula to find other values of the variables
- Analyse and solve problems involving direct and inverse variation

3.1 Direct variation

3.1.1 Directly proportional to

Definition 25

If y is **in proportion** to x (or y **varies proportionately to** x), i.e. $y \propto x$ and

$$y = kx$$

for some $k \in \mathbb{R}$, known as the *constant of proportionality* or *constant of variation*.

Steps

General steps for approaching questions related to variation

1. Read the question carefully

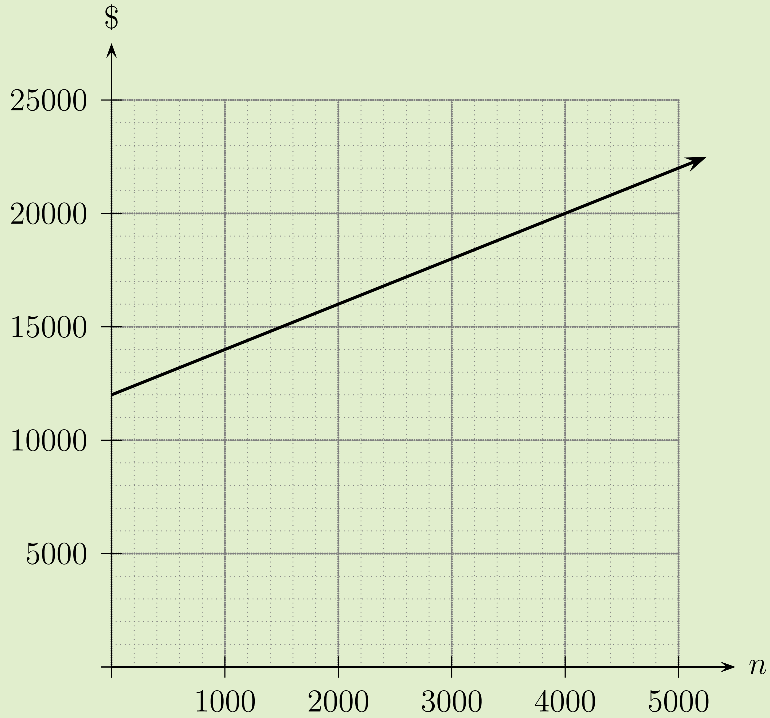
2. Write down the proportionality statement using symbols, then translate to an equation.

3. Find the value of k by substituting values provided.

4. Solve the problem by making another set of substitutions with the value of k found.

Example 42

[2024 Hurlstone Agricultural HS Std 2 Trial Q17] A supermarket chain is considering producing a new biscuit. The cost to produce each packet of biscuits is shown on the graph below, where the vertical axis represents the cost, in dollars, and the horizontal axis represents the number of packets (n) produced.



(a) Determine the fixed cost to produce the biscuits and the production cost per packet. **2**

(b) The revenue from selling the packets of biscuits varies directly with the number of packets sold. When 2000 packets are sold, the revenue is $12 000. **3**

On the grid, graph a line representing the revenue from selling the packets of biscuits and determine the break-even point for the supermarket. **Answer:** 3 000 packets

3.1.2 Directly proportional to x^n

Definition 26

If y is **proportional to the square of** x, or y **varies with** x^2, i.e. $y \propto x^2$ and

$$y = kx^2$$

for some $k \in \mathbb{R}$, known as the *constant of proportionality* or *constant of variation*.

Definition 27

If y is **proportional to** x^n, i.e. $y \propto x^n$ and

$$y = kx^n$$

for some $k \in \mathbb{R}$, known as the *constant of proportionality* or *constant of variation*.

Example 43

[2023 Std 2 HSC Q22] The braking distance of a car, in metres, is directly proportional to the square of its speed in km/h, and can be represented by the equation

$$\text{braking distance } = k \times (\text{speed})^2$$

where k is the constant of variation.

(a) Find the value of k. **2**

(b) What is the braking distance when the speed of the car is 90 km/h? **1**

Answer: (a) $k = 0.08$ (b) Braking dist $= 64.8\,\text{m}$

Example 44

[2008 General HSC Q19] The height of a particular termite mound is directly proportional to the square root of the number of termites.

The height of this mound is 35 cm when the number of termites is 2 000.

What is the height of this mound, in centimetres, when there are 10 000 termites?

Answer: (B)

(A) 16 (B) 78 (C) 175 (D) 875

3.2 Inverse variation

3.2.1 Inversely proportional to

Definition 28

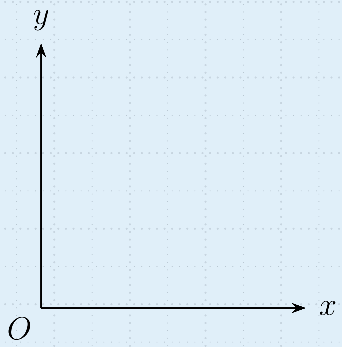
If y is **inversely proportional** to x (or if y **varies inversely with** x), i.e. $y \propto \frac{1}{x}$ and

$$y = \frac{k}{x}$$

for some $k \in \mathbb{R}$, known as the *constant of proportionality* or *constant of variation.*

Fill in the spaces

- Graph of inverse variation $\left(y = \dfrac{k}{x}\right)$:

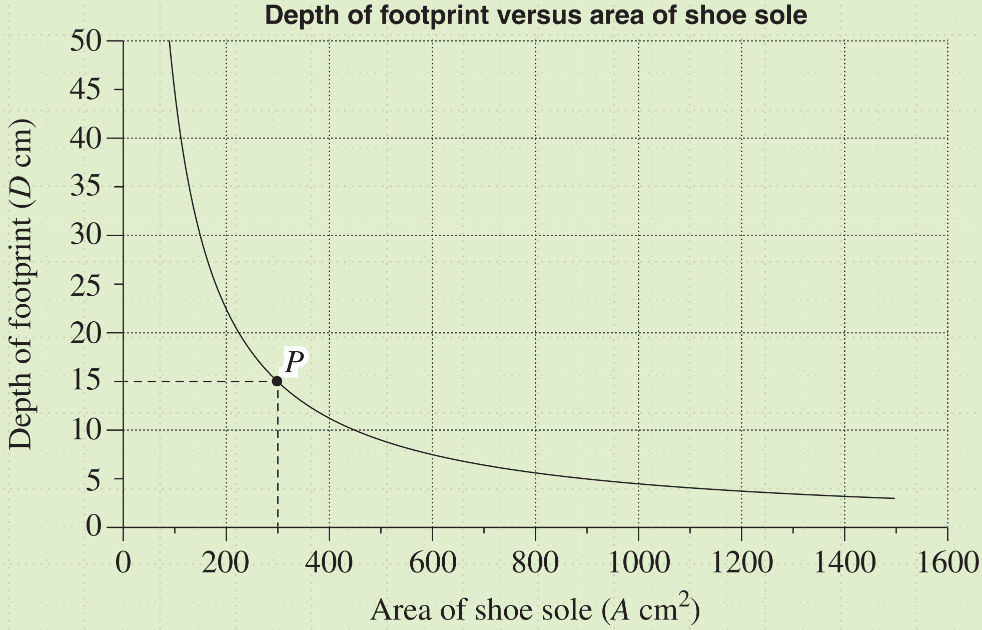


- If two variables are in inverse variation, then as one variable ......increases...... the other ......decreases...... .

Example 45

[2018 General 2 HSC Q29] When people walk in snow, the depth (D cm) of each footprint depends on both the area (A cm^2) of the shoe sole and the weight of the person.

The graph shows the relationship between the area of the shoe sole and the depth of the footprint in snow, for a group of people of the same weight.



(a) The graph is a hyperbola because D is inversely proportional to A. The point P lies on the hyperbola. **2**

Find the equation relating D and A.

(b) A man from this group walks in snow and the depth of his footprint is 4 cm. **1**

Use your equation from part (i) to calculate the area of his shoe sole.

Answer: 1 125 cm^2

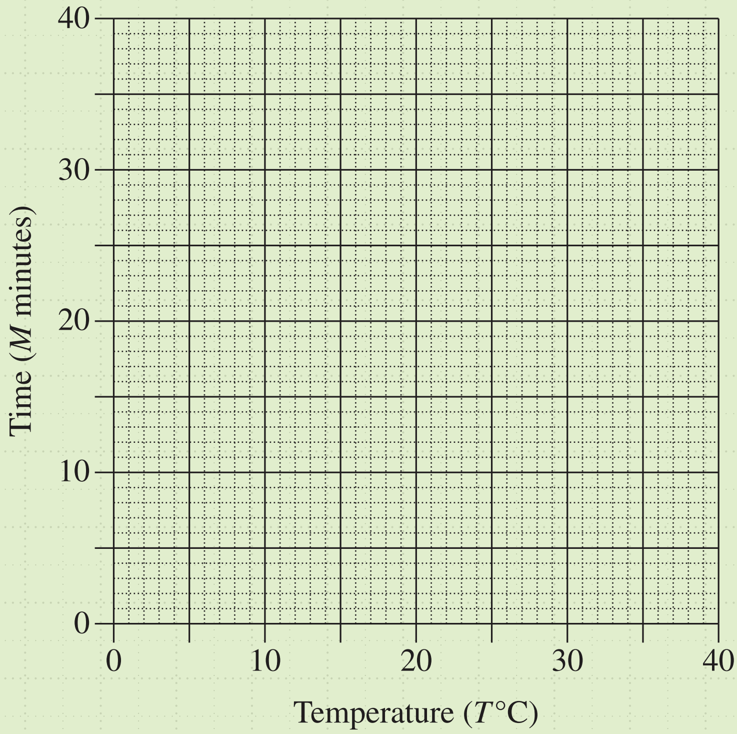
Example 46

[2022 Adv HSC Q12] A student believes that the time it takes for an ice cube to melt (M minutes) varies inversely with the room temperature (T°C). The student observes that at a room temperature of 15°C it takes 12 minutes for an ice cube to melt.

(a) Find the equation relating M and T. **2**

(b) By first completing this table of values, graph the relationship between temperature and time from $T = 5$°C to $T = 30$°C. **2**

T	5	15	30
M			



Example 47

[2023 CSSA Adv Trial Q13] (2 marks) The time it takes to complete a task varies inversely with the number of people assigned.

It takes 5 people to complete a task in 4 hours.

Find the amount of time it would take 8 people to complete the same task.

Answer: 2.5 hrs

3.2.2 Inversely proportional to x^n

Definition 29

If y is **inversely proportional to** x^n, i.e. $y \propto \frac{1}{x^n}$ and

$$y = \frac{k}{x^n}$$

for some $k \in \mathbb{R}$, known as the *constant of proportionality* or *constant of variation*.

Example 48

[1999 MIS HSC Q13] The height of a container is inversely proportional to the square of the radius of its base.

When the height is 5 cm, the radius of the base is 3 cm.

What is the height if the radius of the base is 5 cm? **Answer:** (A)

(A) $1\frac{4}{5}$ cm (B) 3 cm (C) $8\frac{1}{3}$ cm (D) $13\frac{8}{9}$ cm

Example 49

[2027 Adv HSC Sample Q14] (2 marks) The quantity y varies inversely with the square of x.

When $x = 4$, $y = 10$.

Find the value of y when $x = 5$. **Answer:** $x = 6.4$

Further exercises

A **Ex 3G** (Pender et al., 2019a)

- Q12

A **Ex 3H**

- Q7-8, 10-12

X1 **Ex 3G** (Pender et al., 2019b)

- Q13

X1 **Ex 3H**

- Q5-7, 9-11, 13-17

Section 4

Piecewise defined functions & the absolute value

Learning Goal(s)

Knowledge
What absolute value functions are

Skills
Transforming between algebraic and graphical representations, solve absolute value equations

Understanding
Absolute value functions are the reflection of the original graph along the horizontal axis.

By the end of this section am I able to:

2.9 Piecewise-defined functions

- Interpret piecewise-defined functions, where the function is defined differently in different parts of the domain
- Graph piecewise-defined functions involving functions covered in the scope of the Mathematics Advanced course, test if they are even or odd, and determine the domain and range
- Define informally that a function is continuous at a point if the curve can be drawn through the point without lifting the pen off the paper
- Identify points where piecewise-defined functions and other functions are not continuous
- Define a discontinuity of a function informally as a point where the function is not continuous

2.10 Absolute value functions

- Define the absolute value $|x|$ of a number x, also known as the magnitude of x, to be the distance from the origin to x on the number line
- Establish and use the piecewise definition

$$|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0 \end{cases}$$

- Show using numerical substitutions that $\sqrt{x^2} = |x|$ and use the result
- Graph the function $y = |x|$, describe its symmetry, and identify its domain and range
- Graph $y = |ax + b|$ with and without graphing applications, and identify its symmetry, domain and range
- Solve absolute value equations of the form $|ax + b| = k$ algebraically and graphically

4.1 Piecewise defined functions

Definition 30

A *piecewise defined function* is one that has multiple definitions depending on the input values.

Example 50

Sketch $f(x) = \begin{cases} x^2 & x < 0 \\ 3x & 0 \leq x < 4 \\ 12 & x > 4 \end{cases}$.

Example 51

[2023 Independent Adv Trial Q3] A function f is given by the rule

$$f(x) = \begin{cases} (x-2)^2 + 1 & x \leq 2 \\ 4x - 7 & x > 2 \end{cases}$$

What is the value of $f\Big(f(0)\Big)$?

(A) 5 (B) -7 (C) -35 (D) 13

4.2 The absolute value function

Definition 31

The absolute value function

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

Important note

- Geometrically, reflect the ......negative...... parts of the graph about the ..x.. axis.
- Also be ready to see the graph as a transformation.

Example 52

If $f(x) = x^2$ and $g(x) = \sqrt{x}$, sketch $g\big(f(x)\big)$. What observations can be made about $g\big(f(x)\big)$?

Example 53

Sketch $y = |x - 2|$

Example 54

Sketch $y = |2x - 1|$.

Example 55

Sketch $y = |x^2 - 2x - 8|$.

Example 56

[2021 Independent Adv Trial Q2] What is the number of distinct solutions of the equation $|x^2 - 1| = 1$?

(A) 1 (B) 2 (C) 3 (D) 4

Additional questions

1. Sketch each function over the stated domain. State also the range of the function over the specified domain:

(a) $y = 3 - 2x,\ x \geq 1$

(b) $y = x^2,\ 0 \leq x \leq 2$

(c) $xy = 6,\ -2 < x \leq 3,\ x \neq 0.$

(d) $y = (x+2)^2 - 1,\ -3 \leq x \leq 0.$

2. Sketch each of the following piecewise defined functions, showing the coordinates of the endpoints of each interval. State also the range of the function over the specified domain.

(a) $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ -x+1 & \text{if } x \geq 0 \end{cases}$

(b) $f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ 3 & \text{if } x \geq 1 \end{cases}$

(c) $f(x) = \begin{cases} -2-x & \text{if } x \leq 2 \\ 2x-3 & \text{if } x > 2 \end{cases}$

(d) $f(x) = \begin{cases} x^2+1 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$

(e) $f(x) = \begin{cases} -x^2 & \text{if } x < 1 \\ 2^x & \text{if } x > 1 \end{cases}$

(f) $f(x) = \begin{cases} -2x-3 & \text{for } x < -1 \\ -1 & \text{for } -1 \leq x < 1 \\ -\dfrac{1}{x} & \text{for } x \geq 1 \end{cases}$

3. If $f(x) = \begin{cases} 1-x & \text{if } x < 3 \\ x^2+2 & \text{if } x \geq 3 \end{cases}$, evaluate (a) $f(-5)$ (b) $f(3)$ (c) $f(5)$

4. If $f(x) = \begin{cases} x^2-1 & \text{if } x \leq -4 \\ 6 & \text{if } -4 < x < 0 \\ 5x & \text{if } x \geq 0 \end{cases}$, evaluate

(a) $f(-6)$

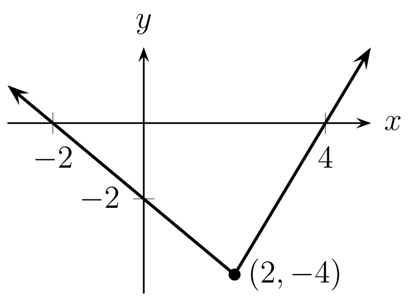
(b) $f(-2)$

(c) $f(-4) + f(2)$

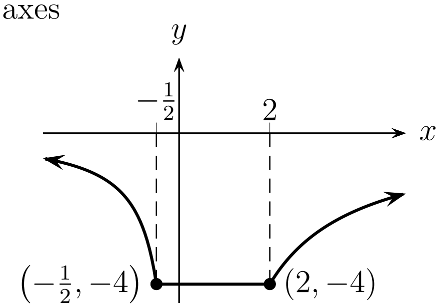
(d) $f(a^2)$

5. Write down the piecewise descriptions for the following functions:

(a)



(b) Both curved sections are hyperbolae whose asymptotes are the x and y axes



6. Sketch the following functions, stating the domain and range in each case:

(a) $y = |x-4|$

(b) $y = -|x+3|$

(c) $y = |x| + 3$

(d) $y = |2x-3|$

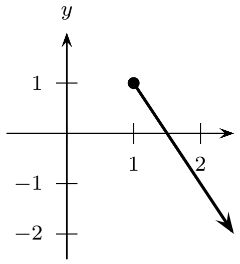
(e) $y = 2|x| - 3$

(f) $y = |3x+7| - 2$

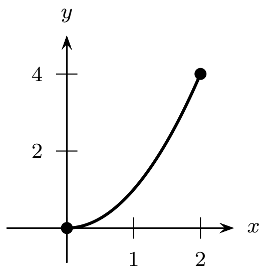
(g) $y = 4 - 3|4-x|$

Answers

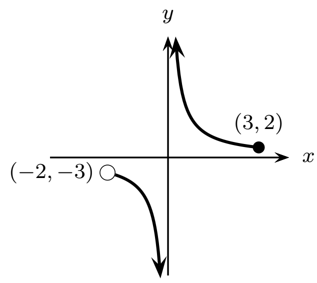
1. (a) $R = \{y : y \leq 1\}$.



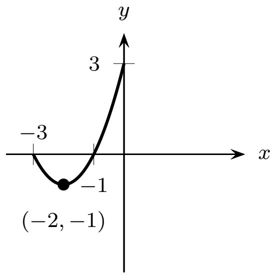
(b) $R = \{y : 0 \leq y \leq 4\}$.



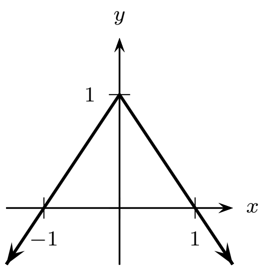
(c) $R = \{y : y < -3, y \geq 2\}$



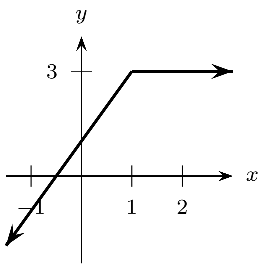
(d) $R = \{y : -1 \leq y \leq 3\}$



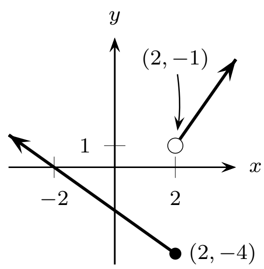
2. (a) $R = \{y : y \leq 1\}$



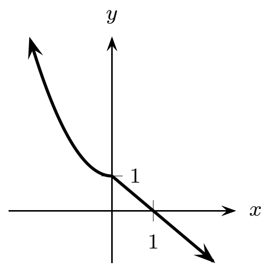
(b) $R = \{y : y \leq 3\}$



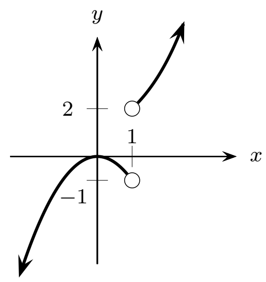
(c) $R = \{y : y \geq -4\}$



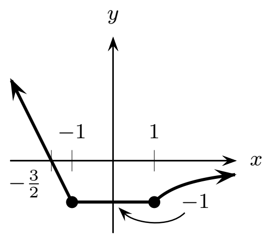
(d) $R = \{y : y \in \mathbb{R}\}$.



(e) $R = \{y : y \leq 0\} \cup \{y : y > 2\}$



(f) $R = \{y : y \geq -1\}$



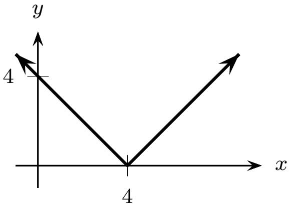
3. (a) 6 (b) 11 (c) 27

4. (a) 35 (b) 6 (c) 25 (d) $5a^2$

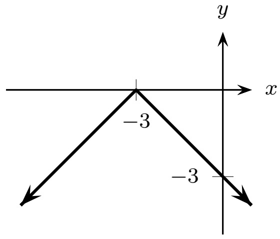
5. (a) $f(x) = \begin{cases} -x-2 & \text{if } x < 2 \\ 2x-8 & \text{if } x \geq 2 \end{cases}$

(b) $f(x) = \begin{cases} \frac{2}{x} & \text{if } x < -\frac{1}{2} \\ -4 & \text{if } -\frac{1}{2} \leq x < 2 \\ -\frac{8}{x} & \text{if } x \geq 2 \end{cases}$

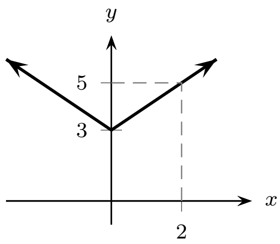
6. (a) $R = \{y : y \geq 0\}$



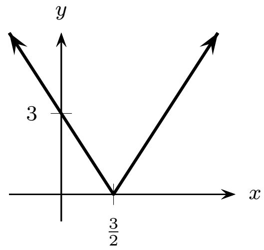
(b) $R = \{y : y \leq 0\}$



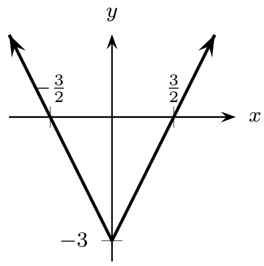
(c) $R = \{y : y \geq 3\}$



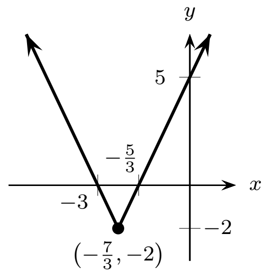
(d) $R = \{y : y \leq 0\}$



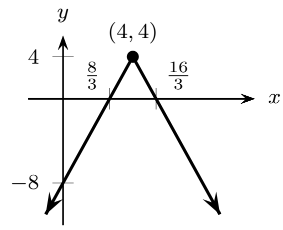
(e) $R = \{y : y \geq -3\}$



(f) $R = \{y : y \geq -2\}$



(g) $R = \{y : y \leq 4\}$



Further exercises

(A) Ex 3B (Pender et al., 2019a)

- Q15

(A) Ex 4D

- Q1, 4, 9-11, 14, 15

(X1) Ex 3A (Pender et al., 2019b)

- Q7 (piecewise defined functions)

(X1) Ex 4D

- Q1, 4, 7, 14, 15, 16, 17, 19

4.2.1 Solving absolute value equations

Important note

- Draw picture!
- When using purely algebraic methods, test values on equations with absolute value on one side and the other side with algebraic terms without absolute value.

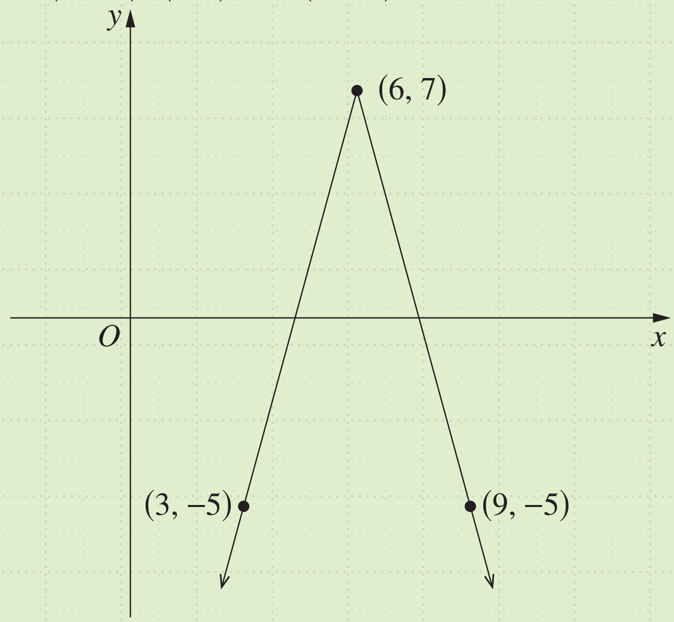
Example 57

Solve $|2x - 1| = 3$.

Answer: $x = -1, 2$

Example 58

[2023 Adv HSC Q27] The graph of $y = f(x)$, where $f(x) = a|x - b| + c$, passes through the points $(3, -5)$, $(6, 7)$ and $(9, -5)$ as shown in the diagram.



(a) Find the values of a, b and c. **3**

(b) The line $y = mx$ cuts the graph of $y = f(x)$ in two distinct places. **2**

Find all possible values of m.

Additional questions

Solve for real values of x:

1. $|x| = 4$

2. $|6x| = 18$

3. $\left|\frac{1}{2}x\right| = 8$

4. $|x-3| = 0$

5. $|x+2| = 12$

6. $|x-6| = 7$

7. $|x+7| = 4$

8. $|x-5| = 2$

9. $|2x-7| = 3$

10. $|3x+4| = 10$

11. $|1-2x| = 5$

12. $|7-3x| = 1$

Answers

1. $x = \pm 4$ **2.** $x = \pm 3$ **3.** $x = \pm 16$ **4.** $x = 3$ **5.** $x = 10, -14$ **6.** $x = 13, -1$ **7.** $x = -3, -11$ **8.** $x = 3, 7$ **9.** $x = 2, 5$ **10.** $x = 2, -\frac{14}{3}$ **11.** $x = 3, -2$ **12.** $x = 2, \frac{8}{3}$

NESA Reference Sheet – Mathematics Advanced

Mathematics Advanced

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}\left(a + b\right)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Financial mathematics

$$FV = PV\left(1 + r\right)^n$$

Sequences and series

$$a_n = a + \left(n - 1\right)d$$

$$S_n = \frac{n}{2}\left[2a + \left(n - 1\right)d\right] = \frac{n}{2}\left(a + a_n\right)$$

$$a_n = ar^{n-1}$$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}, \ r \neq 1$$

$$S_\infty = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and exponential functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Relations

$$\left(x - a\right)^2 + \left(y - b\right)^2 = r^2$$

Trigonometric functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

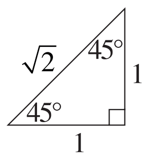
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

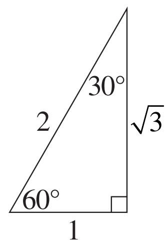
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Differential calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^x$	$\frac{dy}{dx} = (\ln a)a^x$
$y = \log_a x$	$\frac{dy}{dx} = \frac{1}{x \ln a}$

Integral calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int_a^b f(x)\,dx$$

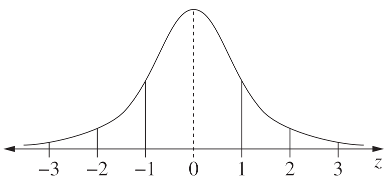
$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Statistical analysis

$$z = \frac{x - \mu}{\sigma}$$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

Discrete random variables

$$\mathrm{E}(X) = \mu$$

$$\mathrm{Var}(X) = \mathrm{E}\left[(X - \mu)^2\right]$$

Continuous random variables

$$P(X \leq x) = F(x) = \int_a^x f(t)\,dt$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

$$\mathrm{E}(X) = \mu = \int_a^b xf(x)\,dx$$

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \mu^2 = \int_a^b x^2 f(x)\,dx - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0$$

References

Coroneos, J. (1990). *A Higher School Certificate Course in Mathematics Year Eleven.* Coroneos Publishing.

Evans, M., Brown, P. G., Gaudry, G., Hunt, D., McLaren, R., Pender, W., & Woolacott, B. (2007). *ICM-EM Mathematics Secondary 4B.* International Centre of Excellence for Education in Mathematics.

McSeveney, A., Conway, R., & Wilkes, S. (1986). *Signpost Mathematics 10 Advanced Course.* Longman Cheshire Pty Ltd.

Pender, W., Sadler, D., Shea, J., & Ward, D. (1999). *Cambridge Mathematics 3 Unit Year 11* (1st ed.). Cambridge University Press.

Pender, W., Sadler, D., Shea, J., & Ward, D. (2009). *Cambridge Mathematics 2 Unit Year 11* (2nd ed.). Cambridge University Press.

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019a). *CambridgeMATHS Stage 6 Mathematics Advanced Year 11* (1st ed.). Cambridge Education.

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019b). *CambridgeMATHS Stage 6 Mathematics Extension 1 Year 11* (1st ed.). Cambridge Education.

Swale, M., Kempton, L., Michell, S., Smith, C., Morris, S., & Rozen, R. (2019). *Jacaranda Maths Quest Mathematical Methods 12 VCE Units 3 and 4* (2nd ed.). John Wiley & Sons Australia, Ltd.