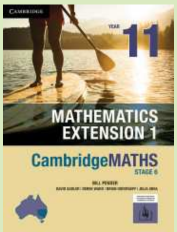
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1
YEAR 11 COURSE

Topic summary and exercises:

With references to



(X1) Further work with functions

Name: ....

Initial version by H. Lam, 2019. Last updated January 18, 2026.
Based on the work from the legacy syllabuses by R. Trenwith, 1995–2010, subsequently maintained by H. Lam, 2011-2026.
Various corrections by students & members of the Department of Mathematics at North Sydney Boys and Normanhurst Boys High Schools.

The questions and structure of document of the *Graphical Techniques* section arise from various sources, including

- *Graphs* Books 1 & 2, P. Marsh.
- *Sketching Graphs, Extension 2*, R. Trenwith.
- Assistance from M. Ziaziaris.
- GeoGebra animations by E. Chiem (Baulkham Hills High School)

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at http://www.flaticon.com, used under CC BY 2.0.

Symbols used

Beware! Heed warning.

Ⓐ Mathematics Advanced content, for `MAA` coded class

Ⓧ1 Mathematics Extension 1 content, for `MAX` coded class

Literacy: note new word/phrase.

Facts/formulae to memorise.

On the course Reference Sheet.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Content outcomes addressed

ME1-11-01 Solves problems involving inequalities, functions and their inverses, graphical relationships between functions, and parametric equations

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Further exercises from
  - *Cambridge Year 11 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019)

  will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Learning intentions & outcomes

✓ **Content**

☐ 3.1 Inequalities

- Solve cubic inequalities where the cubic is expressed as a product of linear factors.
- Solve inequalities involving rational expressions with variables in the denominator.
- Solve absolute value inequalities of the form $|ax+b| \geq k$, $|ax+b| \leq k$, $|ax+b| < k$, $|ax+b| > k$, where a, b and k are constants, using algebraic and graphical methods or the characterisation of $|x|$ as the distance of x from the origin.

☐ 3.2 Graphical relationships

- Examine the relationship between the graph of $y = f(x)$ and the graph of $y = \frac{1}{f(x)}$ using graphing applications, and graph $y = \frac{1}{f(x)}$ given $y = f(x)$ in algebraic or graphical form, identifying any vertical and horizontal asymptotes of $y = f(x)$ and $y = \frac{1}{f(x)}$.
- Examine the relationship between the graphs of $y = f(x)$ and $y = g(x)$ and the graphs of $y = f(x) + g(x)$ and $y = f(x) - g(x)$ using graphing applications, and graph $y = f(x) + g(x)$ and $y = f(x) - g(x)$ given $y = f(x)$ and $y = g(x)$ in algebraic or graphical form.
- Determine the domains and ranges of the sum and difference of functions where possible, and, if appropriate, verify them using a graphing application.
- Examine the relationship between the graph of $y = f(x)$ and the graphs of $y = |f(x)|$ and $y = f(|x|)$ using graphing applications, and graph $y = |f(x)|$ and $y = f(|x|)$ given $y = f(x)$ in algebraic or graphical form.
- Apply knowledge of graphical relationships to solve problems involving graphs of functions, justifying conclusions in the context of the problem where appropriate.

✓ Content

☐ 3.3 Inverse functions

- Describe a function as one-to-one if every element in the range of the function corresponds to exactly one element of the domain.
- Establish that the reflection of a point in the line $y = x$ reverses the coordinates of the point.
- Define an inverse function f^{-1} informally as a function that reverses or undoes the effect of the function f.
- Recognise that inverse functions exist for one-to-one functions.
- Determine the equation for the inverse function $f^{-1}(x)$ of a given one-to-one function $y = f(x)$ by interchanging the variables x and y in $y = f(x)$.
- Compare the graphs of a function and its inverse function using graphing applications, and recognise that the two graphs are reflections of each other in the line $y = x$.
- Establish that the domain of $f^{-1}(x)$ is the range of $f(x)$, and the range of $f^{-1}(x)$ is the domain of $f(x)$, and use this relationship to solve problems.
- Graph the inverse function of a given one-to-one function.
- Explain that the reflection in the line $y = x$ exchanges horizontal and vertical lines and recognise that the horizontal line test can therefore be applied to the graph of $y = f(x)$ to determine whether its reflection in the line $y = x$ is a function.
- Apply restrictions to the domain of a function, if it is not one-to-one, to obtain an inverse function.
- Define formally $f^{-1}(x)$ to be the inverse function of $f(x)$ if the relationships $f\Big(f^{-1}(x)\Big) = x$ and $f^{-1}\Big(f(x)\Big) = x$ hold, and use this definition to solve problems.
- Solve problems based on the relationship between a function and its inverse function using algebraic and graphical techniques, including determining the points of intersection of a function and its inverse, where they exist.

✓ Content

☐ 3.4 Parametric form of a function or relation

- Recognise that a curve may be represented by two parametric equations that give x and y as functions of a parameter and that the curve may also have a Cartesian equation in x and y.
- Convert linear and quadratic functions and circles from parametric form to Cartesian form.
- Express linear and quadratic functions and circles in parametric form.
- Graph linear and quadratic functions and circles expressed in parametric form.

Contents

Part I

Inequalities

Section 1

Linear inequalities

Important note

- The sign of the inequality will ....reverse........ when ....multiplying........ or ....dividing........ by a ....negative........ number.
- Split any double-ended inequalities into two separate ones.

Example 1

Solve $20 > 2 - 3x \geq 8$, sketching the solution on a number line. **Answer:** $x \in (-6, -2]$

Further exercises

Ex 5A

- Q2-3

Section 2

Quadratic, cubic and rational inequalities

2.1 Quadratic

Learning Goal(s)

Knowledge
Solving cubic inequalities

Skills
Determine the range of values satisfying an inequality

Understanding
The need to examine the graph to efficiently solve inequalities

By the end of this section am I able to:

3.1 Inequalities

- Solve cubic inequalities where the cubic is expressed as a product of linear factors.

Important note

- Draw picture! Solving it purely algebraically will definitely cause problems.
- A polynomial inequality should be solved by ....sketching........ the polynomial and solved ....graphically........ .

Example 2

Solve:

(a) $x^2 > 9$

(b) $x + 6 \geq x^2$

Answer: (a) $x \in) -3, 3($ (b) $x \in [-2, 3]$

2.2 Cubic inequalities

Example 3

Sketch the graph of $y = (x+2)(x+1)(x-3)$ and hence find the values of x where

$$(x+2)(x+1)(x-3) < 0$$

Further exercises

Ex 5A

- Q4-6, 9, 13, 16

Ex 5B

- Q1, 4-5, 8

Additional questions

1. Determine the zeros and sketch their graphs, clearly indicating where they cut the x axis:

(a) $y = x(x+2)(x+1)(x-3)$ (c) $y = x(x+1)(x-2)^2$

(b) $y = (x+3)(x+1)(x-1)(3-x)$ (d) $y = (x+1)(x-2)^3$

2. Solve using graphical means:

(a) $(x-2)(x+3) < 0$ (d) $x^2+5x+6 > 0$ (g) $x^3-5x^2+4x \geq 0$

(b) $(x-2)(x+3) > 0$ (e) $x(x+1)(x-1) \leq 0$ (h) $x(x-2)^2 > 0$

(c) $x^2+5x+6 \leq 0$ (f) $-x(x+1)(x-1) \leq 0$ (i) $x(x-2)^2 \leq 0$

Answers

1. Check via GeoGebra or Desmos **2.** (a) $-3 < x < 2$ (b) $x < -3$, $x > 2$ (c) $-3 \leq x \leq -2$ (d) $x < -3$, $x > -2$ (e) $x \leq -1$, $0 \leq x \leq 1$ (f) $-1 \leq x \leq 0$, $x \geq 1$ (g) $0 \leq x \leq 1$, $x \geq 4$ (h) $x > 0$, $x \neq 2$ (i) $x \leq 0$, $x = 2$

2.3 Rational inequalities

Learning Goal(s)

Knowledge
Solving rational expression inequalities

Skills
Identifying when to multiply through by the square of the denominator in inequalities

Understanding
The uncertainty involved in inequalities with unknowns in the denominator

By the end of this section am I able to:

3.1 Inequalities

- Solve inequalities involving rational expressions with variables in the denominator.

Fill in the spaces

An inequality of the form $y = \dfrac{f(x)}{g(x)}$ with an unknown in the denominator will make it uncertain where the function becomes positive or negative.

To avoid this problem,

- multiply by the ......square...... of the denominator, or,
- ......draw...... ......picture...... and solve graphically.

2.3.1 Expressions resulting in a quadratic inequality

Gentle reminder

What is you favourite f-word? ☺

FACTORISE

Example 4

Solve $\dfrac{5}{x-4} \geq 1$

Graphical

Algebraic

Important note

⚠ Remember to exclude a particular value of x!

Example 5

For what values of x is $\dfrac{2x+3}{x-4} > 1$?

Answer: $x < -7$ or $x > 4$.

Graphical

Algebraic

Important note

Remember to exclude a particular value of x!

2.3.2 Expressions resulting in a polynomial inequality

Example 6

Solve $\dfrac{6}{x} \geq x - 5$.

Answer: $0 < x \leq 6$ or $x \leq -1$.

Graphical

Algebraic

Important note

Remember to exclude a particular value of x!

Example 7

[**2002 NEAP Ext 1 Q3**] (3 marks) Solve the inequality $\dfrac{x}{x^2-4} \leq 0$ for x.

Answer: $x \in \{(-\infty, -2) \cup [0, 2)\}$.

Important note

Remember to exclude particular value(s) of x!

Additional questions

Source Portions taken from Fitzpatrick (1984, Ex 23(a))

1. $x^2 - 2x - 15 \leq 0$

2. $x(x-1) \leq 6$

3. $4x^2 - 12x + 10 > 0$

4. $x^2 + 4x + 13 \leq 0$

5. $-3x^2 + 10x + 8 \leq 0$

6. $2x^2 + 5x + 2 \geq 0$

7. $(x-1)(x+3)(x-2) < 0$

8. $(2+x)(x-5)(x+1) > 0$

9. $x^2(x-1) \leq 0$

10. $\dfrac{x-3}{x+1} > 0$

11. $\dfrac{1}{x} > 6$

12. $\dfrac{x-2}{x+3} > -2$

13. $\dfrac{4x-3}{2x+1} \leq 3$

14. $\dfrac{1}{2x-1} \leq 2$

15. $\dfrac{2}{1-x} > -1$

16. $\dfrac{1}{x-2} > 1$

17. $\dfrac{2}{2-x} \geq 3$

18. $\dfrac{1}{x} < \dfrac{1}{4}$

19. $\dfrac{1}{x-3} > 2$

20. $\dfrac{8}{x+1} > 2$

21. $\dfrac{4}{2x-1} < \dfrac{2}{3}$

22. $\dfrac{12}{3x+2} > 4$

23. $\dfrac{6}{5x-2} < 2$

24. $\dfrac{1}{(x-1)(x-3)} \leq -1$

25. $\dfrac{7}{(3-x)(x+3)} > -1$

26. $3x^2 + 5x + 1 > \dfrac{1}{4}$

27. $\dfrac{2x-4}{x+3} > \dfrac{x+2}{2x+6}$

28. $\dfrac{1}{5x^2 - 2x - 7} < \dfrac{5}{13}$

29. $2^{2x} - 5\left(2^x\right) + 4 \leq 0$

30. $2^{2x} - 2\left(2^x\right) \leq -1$

31. ⚠ Find values of x for which the following inequations are simultaneously satisfied:

(a) $\dfrac{x+4}{x+6} < 0$ and $\dfrac{x-6}{x-4} > 1$

(b) $x + \dfrac{1}{|x|} > 0$ and $x^2 - x - 2 < 0$

(c) $x^2 - 5x + 4 \leq 0$ and $6 - x - x^2 > 0$

Answers

1. $-3 \leq x \leq 5$ **2.** $-2 \leq x \leq 3$ **3.** $x \in \mathbb{R}$ **4.** no solution **5.** $x \leq -\frac{2}{3}$ or $x \geq 4$ **6.** $x \leq -2$ or $x \geq -\frac{1}{2}$ **7.** $x < -3$ or $1 < x < 2$ **8.** $-2 < x < -1$ or $x > 5$ **9.** $x \leq 1$ **10.** $x < -1$ or $x > 3$ **11.** $0 < x < \frac{1}{6}$ **12.** $x < -3$ or $x > -\frac{4}{3}$ **13.** $x \leq -3$ or $x > -\frac{1}{2}$ **14.** $x < \frac{1}{2}$ or $x \geq -\frac{4}{3}$ **15.** $x < 1$ or $x > 3$ **16.** $2 < x < 3$ **17.** $\frac{4}{3} \leq x < 2$ **18.** $x < 0$ or $x > 4$ **19.** $3 < x < \frac{7}{2}$ **20.** $-1 < x < 3$ **21.** $x < \frac{1}{2}$ or $x > \frac{7}{2}$ **22.** $-\frac{2}{3} < x < \frac{1}{3}$ **23.** $x < \frac{2}{5}$ or $x > 1$ **24.** $1 < x < 3$ **25.** $x < -4$ or $-3 < x < 3$ or $x > 4$ **26.** $x > -\frac{1}{6}$ or $x < -\frac{3}{2}$ **27.** $x < -3$ or $x > \frac{10}{3}$ **28.** $x < -\frac{6}{5}$ or $-1 < x < \frac{7}{5}$ or $x > \frac{8}{5}$ **29.** $0 \leq x \leq 2$ **30.** $x = 0$ **31.** (a) $-6 < x < -4$ (b) $-1 \leq x < 2$ (c) $-1 < x < 0$ or $0 < x < 2$

Further exercises

Ex 5A

- Q11, 14

Ex 5B

- Q7, 10

Section 3

Absolute value inequalities

Learning Goal(s)

Knowledge
Solving absolute value inequalities

Skills
Identifying when to multiply through by the square of the denominator in inequalities

Understanding
The uncertainty involved in inequalities with unknowns in the denominator

By the end of this section am I able to:

3.1 Inequalities

- Solve absolute value inequalities of the form $|ax+b| \geq k$, $|ax+b| \leq k$, $|ax+b| < k$, $|ax+b| > k$, where a, b and k are constants, using algebraic and graphical methods or the characterisation of $|x|$ as the distance of x from the origin.

3.1 Harder absolute value equations

Important note

Draw picture! Solving it purely algebraically may cause problems.

3.1.1 Absolute value with additional variables

Example 8

Solve $|2x-1| = 3$. **Answer:** $x = -1, 2$

Example 9

Solve $|3x - 7| = |2x - 3|$.

Important note

Draw picture to verify the number of solutions.

Example 10

Find t if $t + |t| = 8$

Example 11

Find a if $|a - 3| + a + 3 = 0$

3.1.2 Absolute value by cases

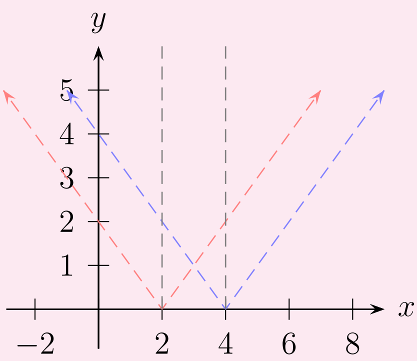
Example 12

(a) ⚠ Sketch $y = |x-2| + |x-4|$.

(b) Hence or otherwise, solve $|x-2| + |x-4| = 4$.

Steps

1. Sketch $y = |x-2|$ and $y = |x-4|$ on the same set of axes:



2. Perform addition of ordinates and take branches to sketch $y = |x-2| + |x-4|$:

- $x < 2$
- $2 < x < 4$
- $x > 4$

(Draw on the same set of axes as previous part)

3. Examine the three 'branches' of $y = |x-2| + |x-4|$:

- $x < 2$:
  - ➋ ...$-$ve... branch $y = |x-2|$
  - ➋ ...$-$ve... branch $y = |x-4|$

  $\therefore y =$ ...$-2x + 6$...

- $x > 4$:
  - ➋ ...$+$ve... branch $y = |x-2|$
  - ➋ ...$+$ve... branch $y = |x-4|$

  $\therefore y =$ ...$2x - 6$...

- $2 < x < 4$:
  - ➋ ...$+$ve... branch $y = |x-2|$
  - ➋ ...$-$ve... branch $y = |x-4|$

  $\therefore y =$ .2.

4. Solve $\begin{cases} y = |x-2| + |x-4| \\ y = 4 \end{cases}$ simultaneously by sketching $y = 4$, and examining intersection with appropriate branches.

Example 13

(a) Sketch $y = |x+1| + |x-2|$.

(b) Hence or otherwise, solve $|x+1| + |x-2| = 6$.

Example 14

Solve $|x-3|+|x+1|=4$.

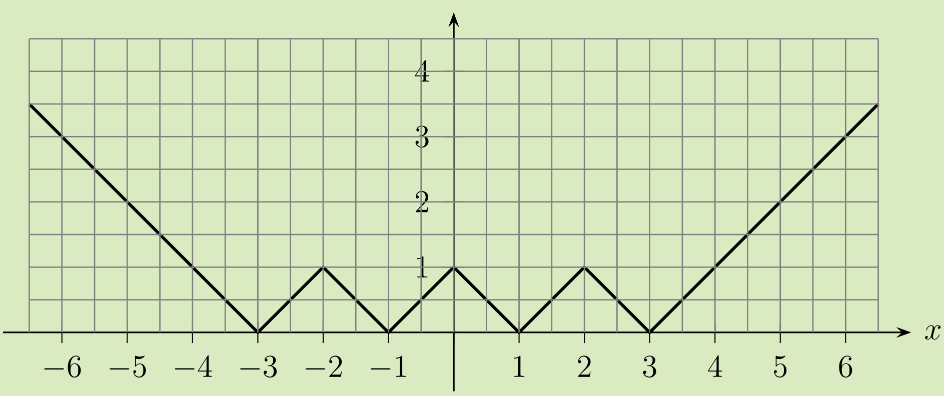
Important note

- What does Example 14 highlight about making an accurate sketch and understanding the graphs?

3.1.3 Nasty looking questions

Example 15

[**2023 Ext 1 HSC Q8**] The diagram shows the graph of a function.



Which of the following is the equation of the function? **Answer:** (A)

(A) $y = \Big|1 - \big||x| - 2\big|\Big|$ (C) $y = \Big|1 - |x - 2|\Big|$

(B) $y = \Big|2 - \big||x| - 1\big|\Big|$ (D) $y = \Big|2 - |x - 1|\Big|$

Important note

The most efficient problem solving technique in this instance: substitute x values

3.2 Solution of inequalities

Example 16

[2012 2U Q11] (2 marks) Solve $|3x - 1| < 2$.

Important note

Which two graphs should be sketched and their points of intersection found in order to verify the solution?

Example 17

(a) Solve $|2x - 5| = x + 2$. **Answer:** $x = 1, 7$

(b) Hence solve $|2x - 5| \geq x + 2$.

Example 18

[2016 NBHS Ext 1 Assessment Task 1] ⚠

(i) On the same set of axes, sketch the graphs of $y = 2|x|$ and $y = |x + 3|$. **4**

(ii) Give the coordinates of the point(s) of intersection of the two graphs. **2**

(iii) Hence solve $2|x| > |x + 3|$. **1**

Example 19

Sketch $y = \dfrac{|x-1|}{(x-1)(x+3)}$.

Example 20

[2013 Ext 1 Q10] Which inequality has the same solutions as

$$|x+2|+|x-3|=5$$

(A) $\dfrac{5}{3-x} \geq 1$

(B) $\dfrac{1}{x-3} - \dfrac{1}{x+2} \leq 0$

(C) $x^2 - x - 6 \leq 0$

(D) $|2x-1| \geq 5$

Example 21

[2025 Independent Ext 1 Trial Q11] Solve for x:

$$\Big||x|-5\Big| \leq 3$$

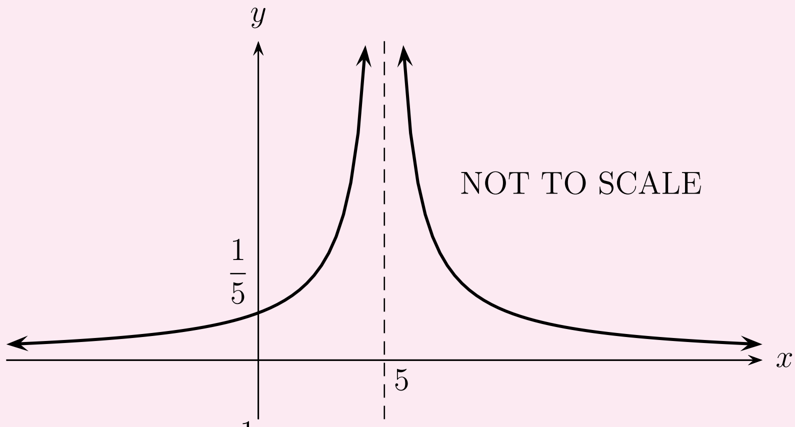
Answer: $x \in [-8,-2] \cup [2,8]$

Important note

$|x| < k$ can also be expressed as $-k < x < k$.

Timed exam practice 1 (Allow approximately 5 minutes)

[2024 Ext 1 HSC Q12] (3 marks) The diagram shows the graph of $y = \dfrac{1}{|x-5|}$.



For what values of x is $\dfrac{x}{6} \geq \dfrac{1}{|x-5|}$?

Marking criteria

- ✓ [1] Recognises the two cases *or* observes that $x = 6$ is a critical point *or* graphs the two curves on the same axis, or equivalent merit.
- ✓ [2] Obtains the three critical points 2, 3 and 6, or equivalent merit.
- ✓ [3] Provides correct solution.

Additional questions

1. Find all solutions to the following:

(a) $|3x-7| = |2x-3|$

(b) $|7-2x| = |x-2|$

(c) $|2+x| = |-x|$

(d) $|3x-1| = |5+2x|$

(e) $|x+1| = |2x+7|$

(f) $|5-3x| = |x+3|$

(g) $|7x-4| = |3x+16|$

(h) $|9x+2| = |3x-4|$

(i) $|2x+1| = |x-2|$

(j) $2\,|x+8| = 3\,|x+5|$

(k) $5\,|x-7| = |9x+1|$

(l) $|7x-3| = 4\,|x+6|$

2. Solve each of the following and verify that the solutions are valid:

(a) $|2x-1| = x+7$

(b) $|x+7| = 2x-1$

(c) $|2x-11| = 3x-4$

(d) $2x-11 = |3x-4|$

(e) $|5-3x| = x+1$

(f) $5-3x = |x+1|$

(g) $|3x+1| = 2x+4$

(h) $|4x-1| = 2x+7$

(i) $|2x| = 9-x$

(j) $|2x+5| = 3x+9$

(k) $|6x-5| = 5x+27$

(l) $|4-2x| = x-2$

3. Solve the following inequalities and graph each solution on the number line.

(a) $|8x| < 24$

(b) $|x+6| < 8$

(c) $|2x-1| < 5$

(d) $|x| > 3$

(e) $|x+4| > 7$

(f) $|3x-2| > 10$

(g) $|4x+3| > 19$

(h) $|5x-2| < 3$

(i) $|2x+5| < 13$

(j) $|2x+6| \leq 12$

(k) $|7x-3| \geq 4$

(l) $|6x+5| < 1$

(m) $\left|\dfrac{x}{2}\right| < 4$

(n) $\left|\dfrac{x}{3}\right| > 5$

(o) $\left|\dfrac{x-1}{2}\right| < 3$

(p) $\left|\dfrac{x+1}{4}\right| > 1$

(q) $|2x+5| < 6$

(r) $|9x-4| \leq 5$

(s) $|5x-1| > 1$

(t) $\left|\dfrac{2x}{5}\right| \leq 4$

(u) $\left|\dfrac{3x-1}{2}\right| > 1$

(v) $\left|\dfrac{2x+5}{3}\right| < 2$

4. Solve by drawing appropriate sketches and examining branches:

(a) $|x| + |x-4| = 2$

(b) $|x-3| + |x+1| = 5$

(c) $|x-3| + |x+1| = 2$

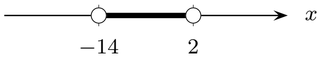
(d) $|x-1| + |x+3| = 4$

Answers

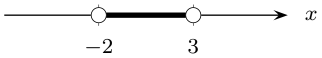
1. (a) $x = 2, 4$ (b) $x = 3, 5$ (c) $x = -1$ (d) $x = 6, -\frac{4}{5}$ (e) $x = -6, -\frac{8}{3}$ (f) $x = 4, \frac{1}{2}$ (g) $x = 5, -\frac{6}{5}$ (h) $x = -1, \frac{1}{6}$ (i) $x = \frac{1}{3}, -3$ (j) $x = 1, -\frac{31}{5}$ (k) $x = -9, \frac{17}{7}$ (l) $x = 9, -\frac{21}{11}$ **2.** (a) $x = 8, -2$ (b) $x = 8$ (c) $x = 3$ (d) no solutions (e) $x = 1, 3$ (f) $x = 1$ (g) $x = -1, 3$ (h) $x = -1, 4$ (i) $x = 3, -9$ (j) $x = -\frac{14}{5}$ (k) $x = -2, 32$ (l) $x = 2$

4. (a) $-3 < x < 3$

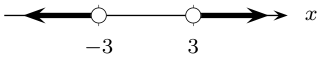
(b) $-14 < x < 2$



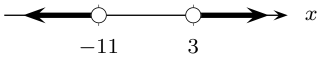
(c) $-2 < x < 3$



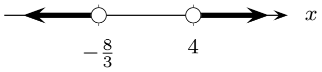
(d) $x > 3$ or $x < -3$



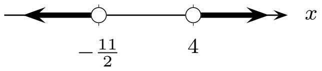
(e) $x > 3$ or $x < -11$



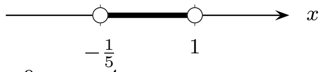
(f) $x > 4$ or $x < -\frac{8}{3}$



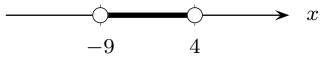
(g) $x > 4$ or $x < -\frac{11}{2}$



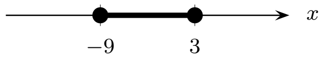
(h) $-\frac{1}{5} < x < 1$



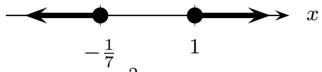
(i) $-9 < x < 4$



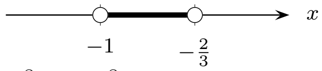
(j) $-9 \le x \le 3$



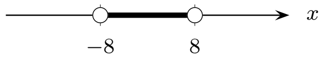
(k) $x \ge 1$ or $x \le -\frac{1}{7}$



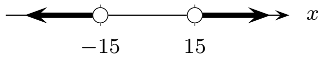
(l) $-1 < x < -\frac{2}{3}$



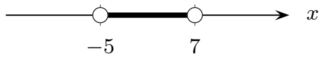
(m) $-8 < x < 8$



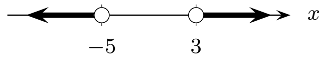
(n) $x > 15$ or $x < -15$



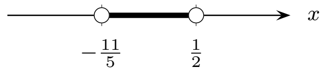
(o) $-5 < x < 7$



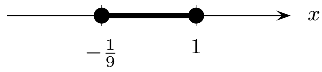
(p) $x > 3$ or $x < -5$



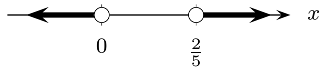
(q) $-\frac{11}{5} < x < \frac{1}{2}$



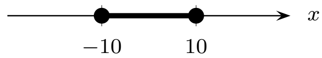
(r) $-\frac{1}{9} \le x \le 1$



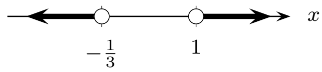
(s) $x > \frac{2}{5}$ or $x < 0$



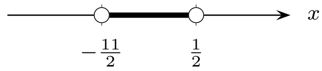
(t) $-10 \le x \le 10$



(u) $x > 1$ or $x < -\frac{1}{3}$



(v) $-\frac{11}{2} < x < \frac{1}{2}$



5. (a) No solution (b) $x = -\frac{3}{2}, \frac{7}{2}$ (c) No solution (d) $-3 \le x \le 1$

Further exercises

Ex 5A

- Q12, 15, 17, 18

Part II

Graphical relationships

Section 4

Reflection about coordinate axes

Learning Goal(s)

Knowledge
Curve sketching techniques

Skills
Applying a curve sketching menu to graph functions

Understanding
The reflection of the graph $y = f(x)$ for the graphs $y = |f(x)|$ and $y = f(|x|)$

By the end of this section am I able to:

3.2 Graphical relationships

- Examine the relationship between the graph of $y = f(x)$ and the graphs of $y = |f(x)|$ and $y = f(|x|)$ using graphing applications, and graph $y = |f(x)|$ and $y = f(|x|)$ given $y = f(x)$ in algebraic or graphical form.

4.1 Basic curve sketching techniques

Steps

1. Determine the ...... domain ....

2. Look for symmetry (... odd ... , even or ...... neither ......).

3. Locate x and y ...... intercepts ....

4. Determine where the graph is ...... positive ...... and where it is ...... negative ....

5. Consider ...... vertical ...... and ...... horizontal ...... asymptotes. Also consider the ...... limiting ...... behaviour as $x \to \pm\infty$.

4.2 Reflection properties

- $f(-x)$: reflection about ..y.. axis.
- $-f(x)$: reflection about ..x.. axis.
- $|f(x)|$: reflect ....negative....... ....ordinates....... about ..x.. axis.
  - Use basic definition of absolute value:

$$|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$$

- $f(|x|)$:
  - Draw sketch of $y = f(x)$ for $x \geq 0$.
  - Reflect about y axis.
  - Use basic definition of absolute value:

$$f(|x|) = \begin{cases} f(x) & x \geq 0 \\ f(-x) & x < 0 \end{cases}$$

4.3 Odd/Even functions

Definition 1

- An *odd* function has .....point..... ....symmetry....... about the .....origin..... .
  Algebraically,

$$f(-x) = -f(x)$$

- An *even* function has ....symmetry....... about the y axis. Algebraically,

$$f(x) = f(-x)$$

Example 22

Sketch the graph of $y = \sqrt{-x}$.

Example 23

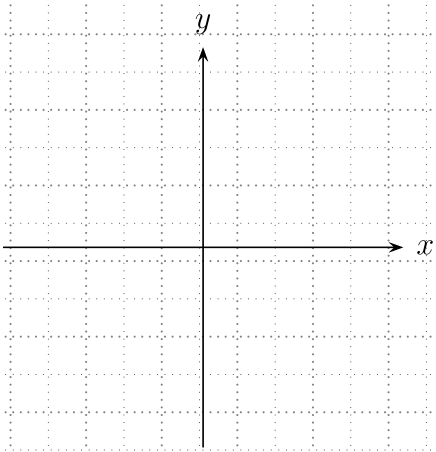
Sketch the graph of $y = |x^2 - 2x|$.

Example 24

For $y = f(x) = (x-2)(x-4)$, sketch:

1. $f(x)$
2. $f(-x)$
3. $-f(x)$
4. $|f(x)|$
5. $f(|x|)$

1.



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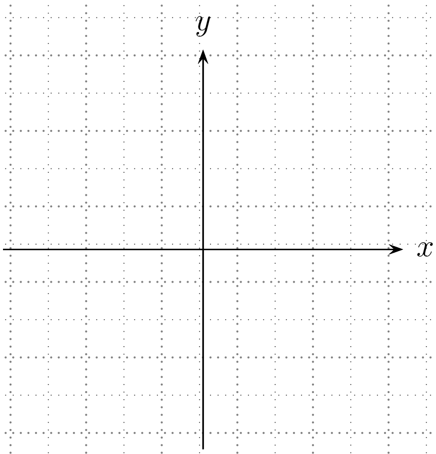

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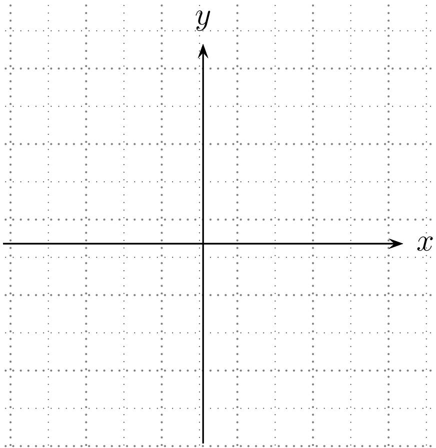

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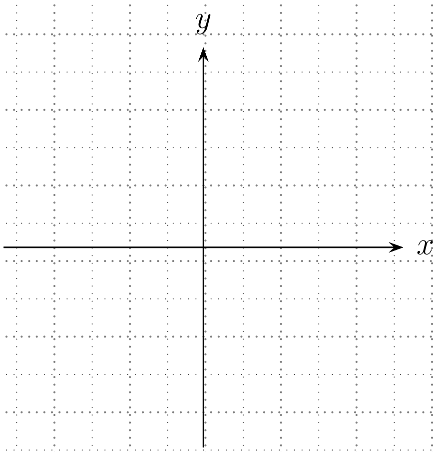

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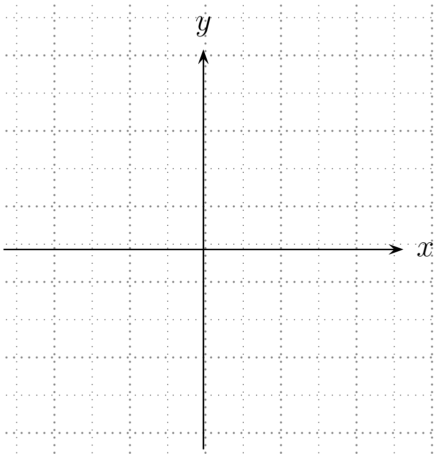

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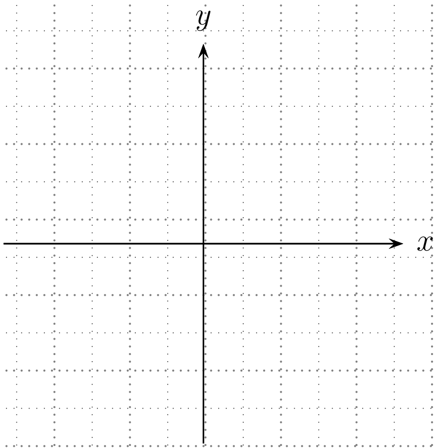


Additional questions

Sketch the following graphs:

1. $y = \dfrac{1}{3^x} - 1$

2. $y = |x(x^2 - 4)|$

3. $y = 5|x| - 2$

4. $y = |x^2| - |x|$

5. $|y| = |x^2 - 1|$

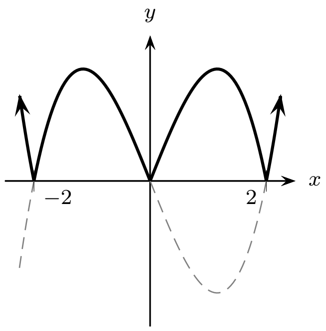
6. $|x| + |y| = 1$

7. $|x + y| = 1$

Answers

1. $y = \dfrac{1}{3^x} - 1$: GeoGebra

2. $y = |x(x^2 - 4)|$



3. $y = 5|x| - 2$: GeoGebra

4. $y = |x^2| - |x|$

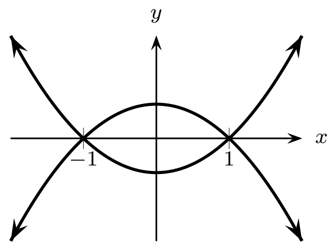
- $x \geq 0$

$$y = x^2 - x$$

- $x < 0$,

$$y = x^2 - (-x)$$

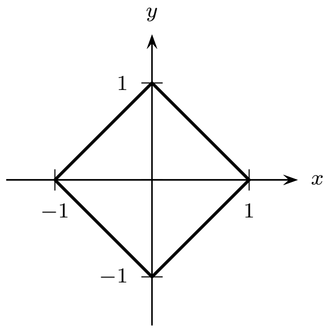
5. $|y| = |x^2 - 1|$



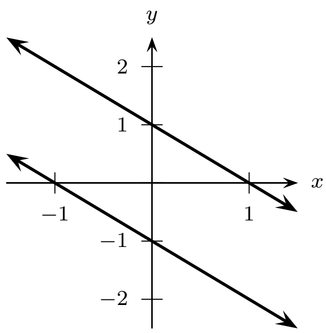
6. $|x| + |y| = 1$
Consider cases:

- $y = 1 - |x|$
- $-y = 1 - |x|$

Also, $x, y < 1$ for this to be true.



7. $|x + y| = 1$



Further exercises

Ex 4B
- Q5-6

Ex 5E
- Q2, 5-6, 8
- Q10, 14

Section 5

Addition/subtraction of ordinates: $y = f(x) \pm g(x)$

Learning Goal(s)

Knowledge
Curve sketching techniques

Skills
Applying a curve sketching menu to graph functions

Understanding
The addition/subtraction of ordinates is via the addition/subtraction of the y values of $f(x)$ and $g(x)$ at the same x value

By the end of this section am I able to:

3.2 Graphical relationships

- Examine the relationship between the graphs of $y = f(x)$ and $y = g(x)$ and the graphs of $y = f(x) + g(x)$ and $y = f(x) - g(x)$ using graphing applications, and graph $y = f(x) + g(x)$ and $y = f(x) - g(x)$ given $y = f(x)$ and $y = g(x)$ in algebraic or graphical form.
- Determine the domains and ranges of the sum and difference of functions where possible, and, if appropriate, verify them using a graphing application.

Steps

1. Ensure that the equation is written in the form $y = f(x) + g(x)$.
   - Test for even/odd function.
2. Find x and y intercepts for $y = f(x) + g(x)$.
3. Draw dotted sketches of $y = f(x)$ and $y = g(x)$ on the same set of axes.
4. Draw perpendicular lines to the x axis that cut $y = f(x)$ and $y = g(x)$.
5. Join points to represent graph of $y = f(x) + g(x)$.

GeoGebra

`Adding-trig-curves.ggb`

Example 25

Sketch $y = x + \dfrac{1}{x}$.

Example 26

Sketch $y = x - \dfrac{1}{x}$.

Example 27

Sketch $y = \sqrt{x-1} + \sqrt{5-x}$.

Additional questions

Sketch the following graphs.

1. $y = x + \dfrac{1}{x^2}$

2. $y = x - \dfrac{1}{x^2}$

3. $y = \dfrac{e^x + e^{-x}}{2}$ $(e \approx 2.71828\cdots)$

4. $y = \dfrac{e^x - e^{-x}}{2}$

5. $y = \dfrac{1}{x + e^x}$

6. $y = \dfrac{1}{x^2 - \frac{1}{x}}$

7. (a) Show that

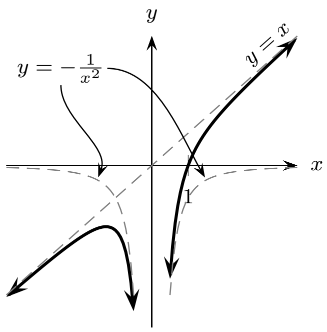
$$\frac{x^3 - x^2 + 1}{x - 1} = x^2 + \frac{1}{x - 1}$$

(b) Hence sketch $y = \dfrac{x^3 - x^2 + 1}{x - 1}$.

Answers

1. $y = x + \dfrac{1}{x^2}$: GeoGebra

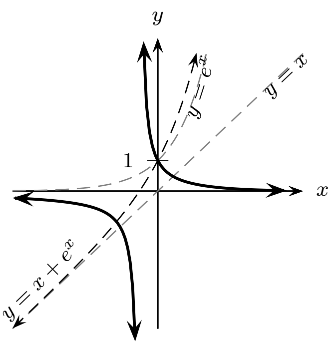
2. $y = x - \dfrac{1}{x^2}$



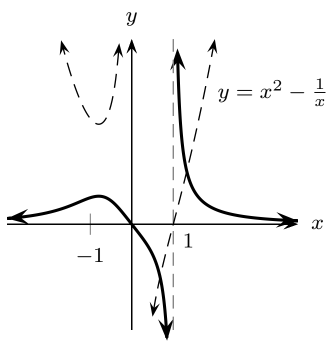
3. $y = \dfrac{e^x + e^{-x}}{2}$: GeoGebra

4. $y = \dfrac{e^x - e^{-x}}{2}$: GeoGebra

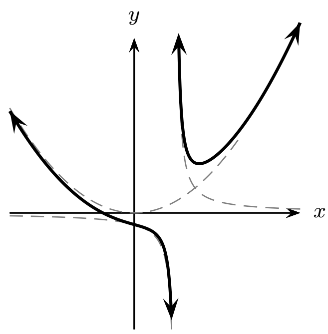
5. $y = \dfrac{1}{x + e^x}$



6. $y = \dfrac{1}{x^2 - \frac{1}{x}}$



7. $y = \dfrac{x^3 - x^2 + 1}{x - 1}$



Section 6

Reciprocal graphs

Learning Goal(s)

Knowledge
Curve sketching techniques

Skills
Applying a curve sketching menu to graph functions

Understanding
$f(x) = 0$ generates vertical asymptotes, and $\lim_{x\to\pm\infty} \frac{1}{f(x)}$ gives the limiting behaviour.

By the end of this section am I able to:

3.2 Graphical relationships

- Examine the relationship between the graph of $y = f(x)$ and the graph of $y = \frac{1}{f(x)}$ using graphing applications, and graph $y = \frac{1}{f(x)}$ given $y = f(x)$ in algebraic or graphical form, identifying any vertical and horizontal asymptotes of $y = f(x)$ and $y = \frac{1}{f(x)}$.

Steps

1. Draw $f(x)$ (draft)

2. Vertical asymptotes whenever $f(x) = 0$ as $y = \dfrac{1}{f(x)}$ does not exist.

3. Find the y intercept of $y = \dfrac{1}{f(x)}$.

4. Draw lines $y = \pm 1$, as $f(x)$ will pass through these exact same points.

5. Consider magnitude of $y = f(x)$ and hence magnitude of $y = \dfrac{1}{f(x)}$.

- As $x \to \pm\infty$.
- As x approaches asymptotes.

Important note

A maximum value on $f(x)$ will result in a ......minimum...... value on $\dfrac{1}{f(x)}$, and vice versa.

⚠ Care is needed for graphs such as $y = \tan x$ and $y = \cot x$.

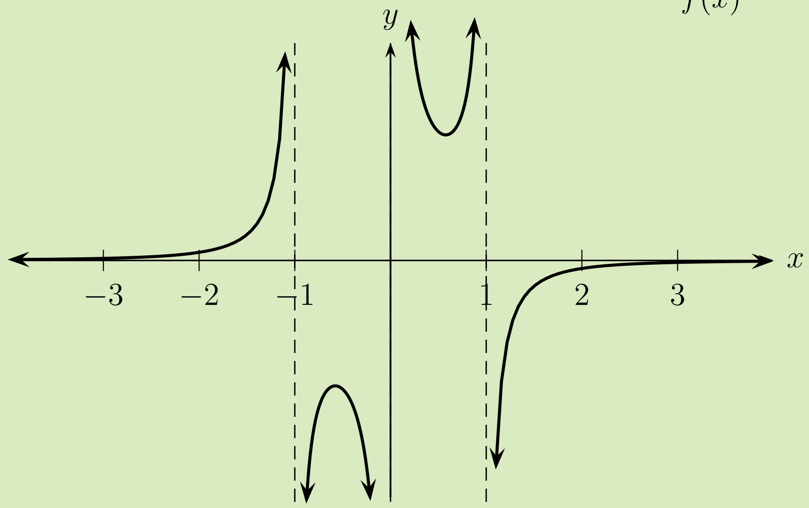
⚠ Later after the *Trigonometry* topic has been taught, when sketching $y = \dfrac{1}{\tan x}$ will take the shape of $y = \cot x$, except when $\tan x = 0$.

Example 28

Sketch $y = \dfrac{1}{(x-1)(x+1)(x-2)}$.

Example 29

[2022 NSGHS Ext 1 Trial Q7] The graph below shows $y = \dfrac{1}{f(x)}$.



Which of the following best represents the equation of $f(x)$? **Answer:** (C)

(A) $f(x) = 1 - x^2$

(B) $f(x) = x\,(x^2 - 1)$

(C) $f(x) = x\,(1 - x^2)$

(D) $f(x) = x^2\,(x^2 - 1)$

6.1 Other manipulations involving $\dfrac{1}{f(x)}$

Laws/Results

Functions of the form $\dfrac{f(x)}{g(x)}$, $g(x) \neq 0$, follow the asymptotes.

Example 30

Sketch $y = \dfrac{x-1}{x-4}$, showing all important features.

Example 31

Sketch $y = \dfrac{2x^2}{x^2-9}$, showing all important features.

Example 32

Sketch $y = \dfrac{1}{x-2} + \dfrac{1}{x-8}$

Example 33

Sketch $y = x^2 + \dfrac{1}{x^2}$.

Example 34

[Ex 3G Q17(a)] ⚠ Show that $\dfrac{(x-1)(x+2)}{x-3} = x+4+\dfrac{10}{x-3}$, and deduce that $y = \dfrac{(x-1)(x+2)}{x-3}$ has an oblique asymptote $y = x+4$. Then sketch the graph.

Example 35

[Ex 3G Q17(b)] ⚠ Likewise, sketch $y = \dfrac{x^2-4}{x+1} = x-1-\dfrac{3}{x+1}$, showing the oblique asymptote.

Additional questions

1. Sketch the following rectangular hyperbolae on separate number planes, stating the domain and range.

(a) $y = \dfrac{1}{x}$

(b) $y = \dfrac{3}{x-3}$

(c) $y = \dfrac{-3}{x-3}$

(d) $y = 1 + \dfrac{1}{x+2}$

(e) $y = 4 - \dfrac{3}{x-3}$

(f) $y = \dfrac{2}{x-4} + 1$

(g) $y = -2 - \dfrac{1}{x+1}$

(h) $y = \dfrac{1}{2x-3} - 3$

(i) $y = \dfrac{x+2}{x+1}$

(j) $y = \dfrac{x}{x+1}$

(k) $y = \dfrac{2x+1}{x-2}$

(l) $y = \dfrac{3x-2}{x+3}$

(m) $y = \dfrac{1-x}{1+x}$

2. Reciprocal functions

(a) i. On the same set of axes, sketch the graphs of $y = x$ and $y = -\dfrac{1}{x}$.

ii. By addition of ordinates, sketch the graph of $y = x - \dfrac{1}{x}$.

(b) Sketch the following by the addition of ordinates:

i. $y = 2x + \dfrac{1}{x}$

ii. $y = \dfrac{3}{x} - 2x$

iii. $y = x^2 + \dfrac{1}{x}$

iv. $y = x^2 - \dfrac{1}{x}$

(c) Sketch the following graphs of reciprocal functions, stating the domain and range:

i. $y = \dfrac{1}{x^2}$

ii. $y = \dfrac{1}{(x-2)^2}$

iii. $y = \dfrac{1}{x^3 - x}$

iv. $y = \dfrac{2}{x^2+1}$

v. $y = \dfrac{2}{x^2-1}$

vi. $y = \dfrac{1}{\sqrt{4-x^2}}$

vii. $y = \dfrac{2}{(x-3)(x+2)}$

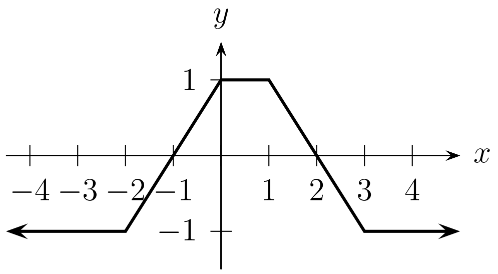
viii. $y = \dfrac{1}{x^2+x}$

ix. $y = \dfrac{x+1}{(x-1)(x+2)}$

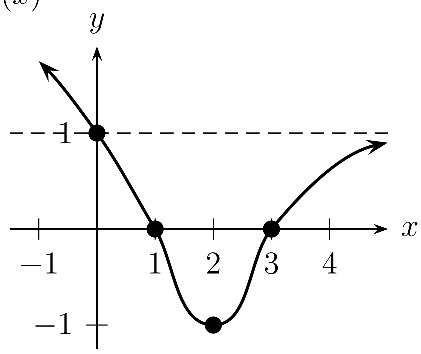
x. $y = \dfrac{x^2-4}{x^2+2x-3}$

3. $f(x)$ is shown in the following diagrams. Sketch $\dfrac{1}{f(x)}$.

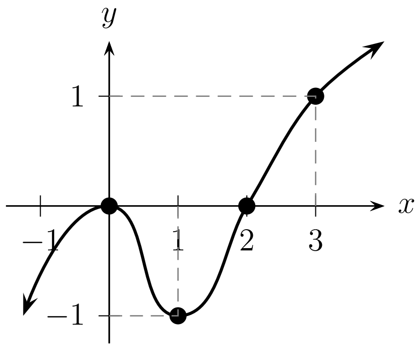
(i)



(iii)



(ii)



4. Make neat sketches of the following graphs:

(a) $y = 2 - x$ and $y = \dfrac{1}{2-x}$

(b) $y = 3x + 1$ and $y = \dfrac{1}{3x+1}$

(c) $y = x^2 - 4$ and $y = \dfrac{1}{x^2-4}$

(d) $y = 9 - x^2$ and $y = \dfrac{1}{9-x^2}$

(e) $y = x^2 - x - 2$ and $y = \dfrac{1}{x^2-x-2}$

(f) $y = \sqrt{x}$ and $y = \dfrac{1}{\sqrt{x}}$

(g) $y = |x|$ and $y = \dfrac{1}{|x|}$

(h) $y = 2^x$ and $y = \dfrac{1}{2^x}$

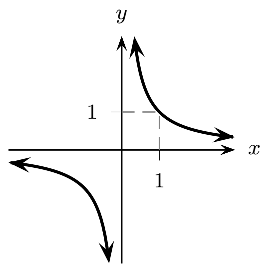
(i) $y = \sin x$ and $y = \dfrac{1}{\sin x}$

(j) $y = (x-1)\left(x^2 - 3x - 4\right)$ and $y = \dfrac{1}{(x-1)\left(x^2-3x-4\right)}$

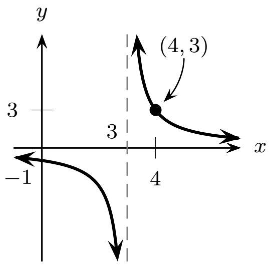
(k) $y = |2x + 1|$ and $y = \dfrac{1}{|2x+1|}$

Answers

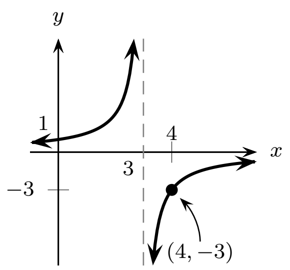
1. (a) $D = \{x : x \neq 0\}$
$R = \{y : y \neq 0\}$



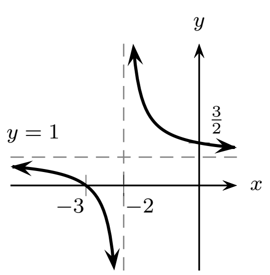
(b) $D = \{x : x \neq 3\}$
$R = \{y : y \neq 0\}$



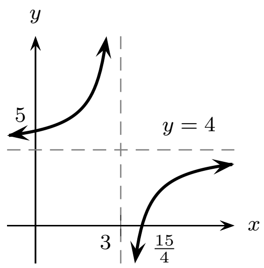
(c) $D = \{x : x \neq 3\}$
$R = \{y : y \neq 0\}$



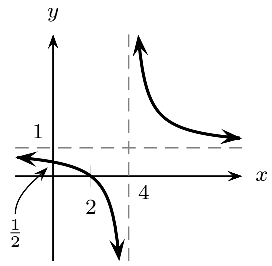
(d) $D = \{x : x \neq -2\}$
$R = \{y : y \neq 1\}$



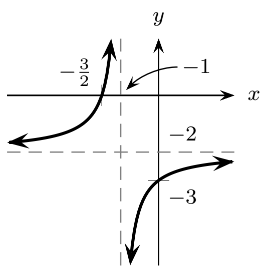
(e) $D = \{x : x \neq 3\}$
$R = \{y : y \neq 4\}$



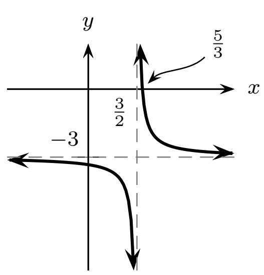
(f) $D = \{x : x \neq 4\}$
$R = \{y : y \neq 1\}$



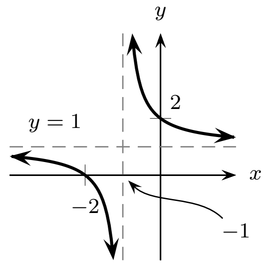
(g) $D = \{x : x \neq -1\}$
$R = \{y : y \neq -2\}$



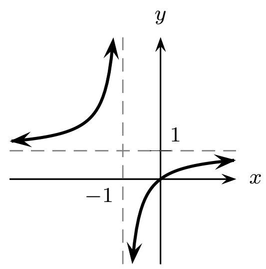
(h) $D = \left\{x : x \neq \frac{3}{2}\right\}$
$R = \{y : y \neq -3\}$



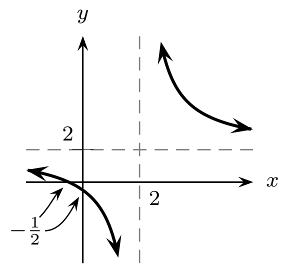
(i) $D = \{x : x \neq -1\}$
$R = \{y : y \neq 1\}$



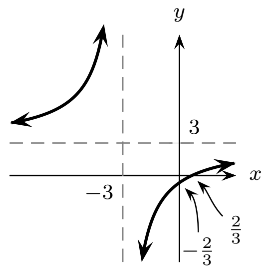
(j) $D = \{x : x \neq -1\}$
$R = \{y : y \neq 1\}$



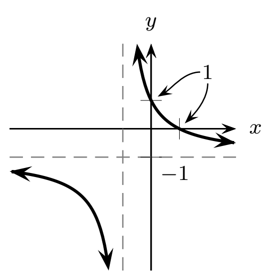
(k) $D = \{x : x \neq 2\}$
$R = \{y : y \neq 2\}$



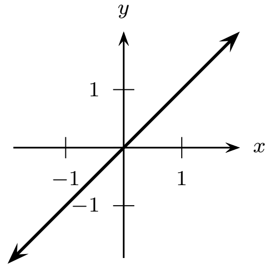
(l) $D = \{x : x \neq -3\}$
$R = \{y : y \neq 3\}$



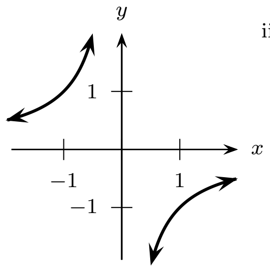
(m) $D = \{x : x \neq -1\}$
$R = \{y : y \neq -1\}$



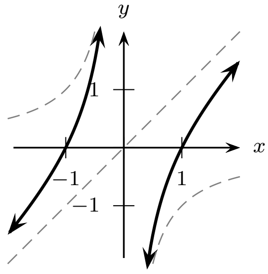
2. (a)



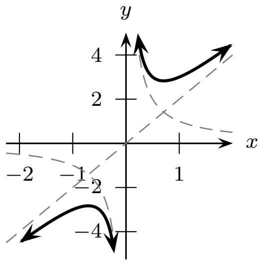
i.



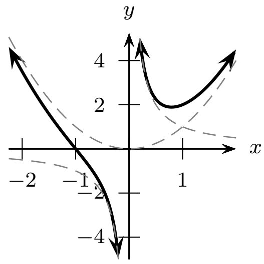
ii.



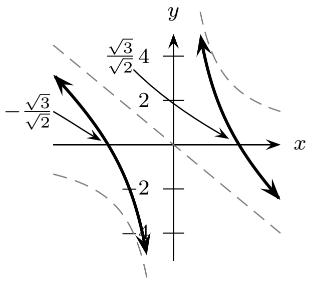
(b) i. $y = 2x + \dfrac{1}{x}$



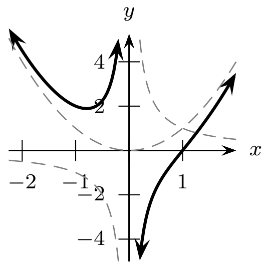
iii. $y = x^2 + \dfrac{1}{x}$



ii. $y = \dfrac{3}{x} - 2x$



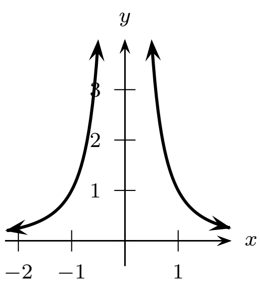
iv. $y = x^2 - \dfrac{1}{x}$

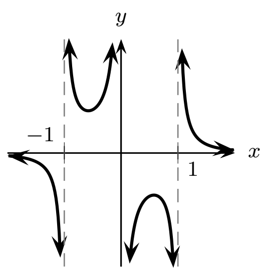


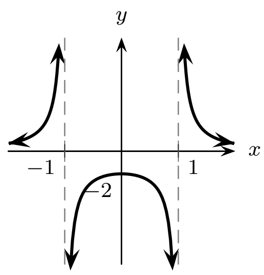
(c) i. $D = \{x : x \neq 0\}$
$R = \{y : y > 0\}$

iii. $D = \{x : x \neq 0, \pm 1\}$
$R = \{y : y \neq 0\}$

v. $D = \{x : x \neq \pm 1\}$
$R = \{y : y \leq -2, y > 0\}$



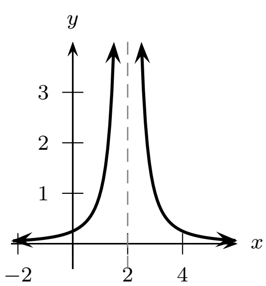


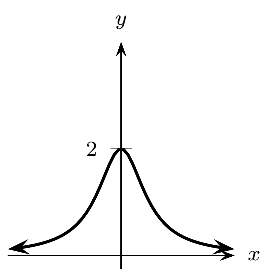


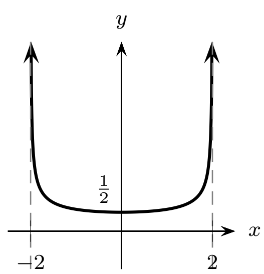
ii. $D = \{x : x \neq 2\}$
$R = \{y : y > 0\}$

iv. $D = \{x : x \in \mathbb{R}\}$
$R = \{y : y > 0\}$

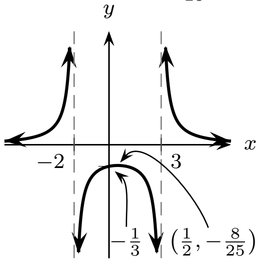
vi. $D = \{x : -2 < x < 2\}$
$R = \left\{y : y > \frac{1}{2}\right\}$



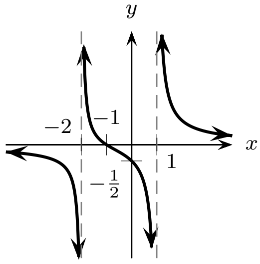




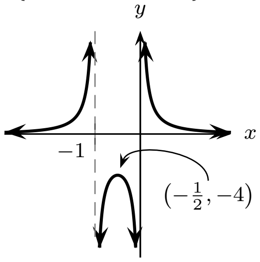
vii. $D = \{x : x \neq 3, -2\}$
$R = \left\{y : y > 0, y \leq -\frac{8}{25}\right\}$



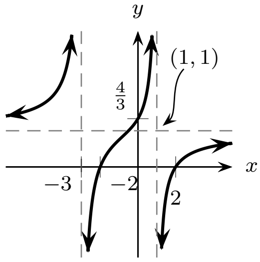
ix. $D = \{x : x \neq 1, -2\}$
$R = \{y : y \in \mathbb{R}\}$



viii. $D = \{x : x \neq -1, 0\}$
$R = \{y : y \leq -4, y > 0\}$



x. $D = \{x : x \neq -3, 1\}$
$R = \{y : y \in \mathbb{R}\}$



3. Remaining exercises: check GeoGebra where possible.

Further exercises

Ex 5C

- Q5-12, 14

Part III

Inverse functions and parametric representation

Section 7

Inverse functions

Learning Goal(s)

Knowledge
Properties of inverse functions

Skills
Determining if a function has an inverse function

Understanding
The difference between inverse relations and inverse functions

By the end of this section am I able to:

3.3 Inverse functions

- Describe a function as one-to-one if every element in the range of the function corresponds to exactly one element of the domain.
- Establish that the reflection of a point in the line $y = x$ reverses the coordinates of the point.
- Define an inverse function f^{-1} informally as a function that reverses or undoes the effect of the function f.
- Recognise that inverse functions exist for one-to-one functions.
- Determine the equation for the inverse function $f^{-1}(x)$ of a given one-to-one function $y = f(x)$ by interchanging the variables x and y in $y = f(x)$.
- Compare the graphs of a function and its inverse function using graphing applications, and recognise that the two graphs are reflections of each other in the line $y = x$.
- Establish that the domain of $f^{-1}(x)$ is the range of $f(x)$, and the range of $f^{-1}(x)$ is the domain of $f(x)$, and use this relationship to solve problems.
- Graph the inverse function of a given one-to-one function.
- Explain that the reflection in the line $y = x$ exchanges horizontal and vertical lines and recognise that the horizontal line test can therefore be applied to the graph of $y = f(x)$ to determine whether its reflection in the line $y = x$ is a function.
- Apply restrictions to the domain of a function, if it is not one-to-one, to obtain an inverse function.
- Define formally $f^{-1}(x)$ to be the inverse function of $f(x)$ if the relationships $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$ hold, and use this definition to solve problems.
- Solve problems based on the relationship between a function and its inverse function using algebraic and graphical techniques, including determining the points of intersection of a function and its inverse, where they exist.

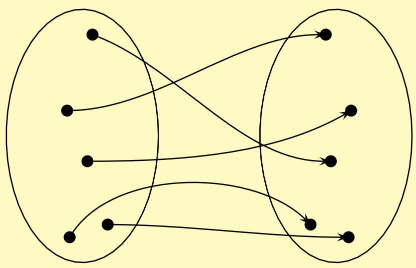
7.1 Conditions for existence of an inverse function

Definition 2

An *inverse relation* to $f(x)$ is denoted $f^{-1}(x)$. It "undo"s whatever $f(x)$ would do.

Laws/Results

A function only has an inverse if and only if it is one to one .



⚠ Functions/relations will always have inverse *relations*.

⚠ "Inverse" almost always implies "inverse function ", not inverse relation .

Definition 3

Monotonic functions a one to one function is also *monotonic increasing* (or *monotonic decreasing*): if $b > a$, then $f(b) > f(a)$.

Important note

⚠ Asymptotes *may* cause functions to not be monotonic. Varies from case to case!

Example 36

Which of the following functions are monotonic?

1. $y = 2x$ **2.** $y = x^2$ **3.** $y = \frac{2x}{x+2}$ **4.** $y = x - \frac{1}{x}$

Additional questions

Source Fitzpatrick (1984, Ex 26(a))

1. State whether the following are one-to-one functions.

(a) $f(x) = x - 2,\ x \in \mathbb{R}$

(b) $f(x) = x^2 - 4x + 1,\ x \geq 2$

(c) $f(x) = \sqrt{4 - x^2},\ x \in [-2, 2]$

(d) $f(x) = 9 - x,\ x \in \mathbb{R}$

(e) $f(x) = \dfrac{1}{9 - x},\ x \neq 9$

(f) $f(x) = |9 - x|,\ x \in \mathbb{R}$

(g) $f(x) = 9 - x^2,\ x \in \mathbb{R}$

(h) $f(x) = x^3 - 4x,\ x \in [-2, 2]$

2. Find the largest possible domain for which the following are one-to-one (monotonic increasing) functions.

(a) $f(x) = \sqrt{4 - x^2}$

(b) $f(x) = \sqrt{x^2 - 4}$

(c) $f(x) = -\dfrac{1}{x + 2}$

(d) $f(x) = 3x - x^2$

(e) $f(x) = x^2 + 6x + 8$

3. Explain why the following functions do not have an inverse function. Suggest suitable restrictions to their domain so that the restricted function may have an inverse.

(a) $f(x) = \sqrt{a^2 - x^2},\ x \in [-a, a]$

(b) $f(x) = 4 - x^2$

(c) $f(x) = \dfrac{1}{x^2},\ x \neq 0$

Answers

1. (a) Yes (b) Yes (c) No (d) Yes (e) Yes (f) No (g) No (h) No **2.** (a) $x \in [-2, 0]$ (b) $x \geq 2$ (c) $x < -2$ (d) $x \leq \frac{3}{2}$ (e) $x \geq -3$ **3.** These are some of the possible domain restrictions. There are others which will also work. (a) $x \in [0, a]$ (b) $x \in [0, 2]$ (c) $x \in (0, \infty)$

7.2 Properties of functions and their inverses

- Mutual inverse (over ....restricted........ ....domain........)

$$x \to \boxed{f} \xrightarrow{\quad f(x) \quad} \boxed{f^{-1}} \to x \qquad x \to \boxed{f^{-1}} \xrightarrow{\quad f^{-1}(x) \quad} \boxed{f} \to x$$

- Domain/range swap:

- Reflection about $y = x$.
  - $f(x)$ and $f^{-1}(x)$ intersect on ....$y = x$....
  - If f and f^{-1} intersect more than once, at least one intersection is on ....$y = x$....
    (Counterexample: ....$y = -x^3$....)

- $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$ (Property for later after derivatives have been dealt with)

7.3 Finding the inverse relation

Steps

1. Interchange x and y.
2. Change subject to y.

Example 37

Find the inverse function of $y = x + 1$.

Example 38

Find the equations of the inverse relations to these functions. If the inverse is a function, find an expression for $f^{-1}(x)$, and verify that $f^{-1}\Big(f(x)\Big) = x$ and $f\Big(f^{-1}(x)\Big) = x$.

(a) $f(x) = 6 - 2x,\ x > 0.$

(b) $f(x) = x^3 + 2.$

(c) $f(x) = \dfrac{1-x}{1+x}.$

(d) $f(x) = x^2 - 9.$

Example 39

(a) Find the inverse function of $y = \dfrac{2}{x-1}$.

(b) Sketch the inverse on the same number plane as the original function.

Example 40

For the function of $f(x) = \dfrac{x+1}{x-2}$,

Answer: (a) $y = 1 + \frac{3}{x-2}$ (b) $y = 2 + \frac{3}{x+1}$

(a) Explain why $f(x)$ has an inverse function.

(b) Find the equation of the inverse function.

(c) Write down the domain and range of the inverse function.

Example 41

Find the inverse function of $y = x^3 - 2$, and sketch both curves on the same set of axes.

Example 42

For the function $y = x^2 + 2x + 3$:

(a) State its domain and range.

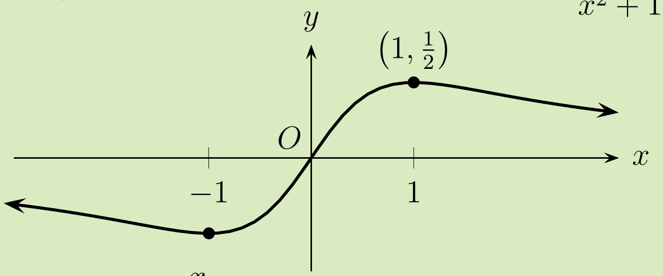
(b) Sketch the curve $y = f(x)$.

(c) Explain why the inverse function does not exist.

(d) Restrict the domain such that the curve is monotonically increasing, sketch the inverse $f^{-1}(x)$ on the same axes as $f(x)$, and find its equation.

Example 43

[2018 Ext 1 HSC Q13] The diagram shows the graph $y = \dfrac{x}{x^2+1}$, for all real x.



Consider the function $f(x) = \dfrac{x}{x^2+1}$, for $x \geq 1$.

The function $f(x)$ has an inverse. (Do NOT prove this.)

i. State the domain and range of $f^{-1}(x)$. **2**

ii. Sketch the graph $y = f^{-1}(x)$. **1**

iii. Find an expression for $f^{-1}(x)$. **3**

Example 44

[1996 3U HSC Q7] Consider the function $f(x) = \frac{1}{4}\left[(x-1)^2 + 7\right]$.

(i) Sketch the parabola $y = f(x)$, showing clearly any intercepts with the axes, and the coordinates of its vertex. Use the same scale on both axes. **2**

(ii) What is the largest domain containing the value $x = 3$, for which the function has an inverse function $f^{-1}(x)$? **1**

(iii) Sketch the graph of $y = f^{-1}(x)$ on the same set of axes as your graph in part (i). Label the two graphs clearly. **2**

(iv) What is the domain of the inverse function? **1**

(v) ⚠ Let a be a real number not in the domain found in part (ii). Find $f^{-1}\Big(f(a)\Big)$. **2**

(vi) Find the coordinates of any points of intersection of the two curves $y = f(x)$ and $y = f^{-1}(x)$. **2**

Further exercises

Ex 5F

- All questions

Ex 5G

- Q1-9

Additional questions

1. Find the inverse function of each of the following functions:

(a) $y = x + 1$

(b) $y = 3x - 2$

(c) $y = \dfrac{x+2}{3}$

(d) $y = x^3$

(e) $y = (x+1)^3$

(f) $y = \dfrac{1}{x-1}$

(g) $y = \dfrac{x}{x-1}$

2. The function $y = x$ is *invariant under inversion*, i.e. the equation of the function and its inverse are the same.

(a) Give two more examples of functions that are invariant under inversion.

(b) What do you notice about the the graphs of these functions?

3. (a) Sketch the graph of $y = \sqrt{3-x}$.

(b) On the same axes, sketch the graph of its inverse function.

(c) Find the equation of the inverse function.

(d) Find the coordinates of the point of intersection of the function and its inverse.

4. Show that the following pairs of functions are inverses by showing that over the restricted domain,

$$f \circ g(x) = g \circ f(x) = x$$

(a) $\begin{cases} f(x) = 2x - 1 \\ g(x) = \dfrac{1}{2}(x+1) \end{cases}$

(b) $\begin{cases} f(x) = \sqrt{16 - x^2} & x \in [-4, 0] \\ g(x) = -\sqrt{16 - x^2} & x \in [0, 4] \end{cases}$

(c) $\begin{cases} f(x) = 2x - x^2 & x \geq 1 \\ g(x) = 1 + \sqrt{1-x} & x \leq 1 \end{cases}$

(d) $\begin{cases} f(x) = \dfrac{1}{2x-1} & x > \dfrac{1}{2} \\ g(x) = \dfrac{x+1}{2x} & x > 0 \end{cases}$

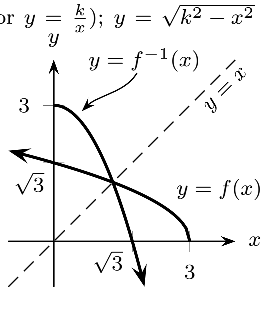
5. **[2024 NBHS Year 11 Ext 1 Task 3 Q12]** **3**

Consider the even function $f(x+a) = x^4$, where $a > 0$. $f^{-1}(x)$ is the inverse function of $f(x)$ and the range of $y = f^{-1}(x)$ is $y \leq a$.

If $c > a$, find the value of $f^{-1}\left[f(c)\right]$ in terms of a and c.

Answers

1. (a) $y = x - 1$ (b) $y = \frac{x+2}{3}$ (c) $y = 3x - 2$ (d) $y = \sqrt[3]{x}$ (e) $y = \sqrt[3]{x} - 1$ (f) $y = 1 + \frac{1}{x}$ (g) $y = \frac{x}{x-1}$ **2.** (a) $y = -x$ (or more generally, $y = k - x$, k being any constant); $y = \frac{1}{x}$ (or $y = \frac{k}{x}$); $y = \sqrt{k^2 - x^2}$ over $D = \{x : 0 \leq x \leq k\}$; plus an infinite number of others. (b) Symmetrical about $y = x$. **3.** (a)



(b) See previous. (c) $y = 3 - x^2$, $x \geq 0$ (d) $\left(\frac{\sqrt{13}-1}{2}, \frac{\sqrt{13}-1}{2}\right)$ **4.** $f^{-1}\Big(f(c)\Big) = 2a - c$

Section 8

Parametric representation

Learning Goal(s)

Knowledge
Parametric form of functions and relations

Skills
Converting Cartesian form to parametric form

Understanding
The additional parameter used to represent the curve in terms of x and y values

By the end of this section am I able to:

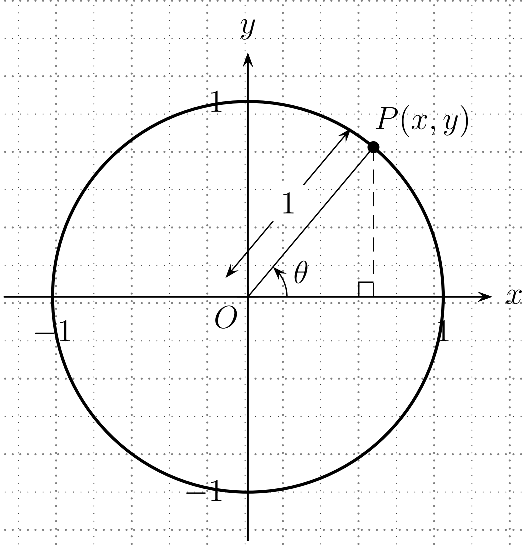
3.21 Parametric form of a function or relation

- Recognise that a curve may be represented by two parametric equations that give x and y as functions of a parameter and that the curve may also have a Cartesian equation in x and y.
- Express linear and quadratic functions and circles in parametric form.
- Convert linear and quadratic functions and circles from parametric form to Cartesian form.
- Graph linear and quadratic functions and circles expressed in parametric form.

8.1 Introduction

Example 45

Write down the equation of a circle with radius 1, centred at the origin in two different forms.



Definition 4

A *Cartesian* equation uses *rectangular coordinates* (x axis $\perp$ y axis). Named after René Descartes (1596–1650). http://en.wikipedia.org/wiki/Rene_Descartes)

- Other coordinate systems include
  - logarithmic $(x, \log_{10} x)$ – Richter scale, dB.
  - X2 polar (r, θ) – navigation, bearings
  - cylindrical (3D polar) (ρ, θ, ϕ) – astronomy, cartography (GPS).

Definition 5

A *parametric* equation uses additional *parameter*(s) to represent the curve in terms of its x and y values.

⚠ There may be more than one parametric representation for the same curve.

8.1.1 Common parameterisations

Circle $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

- r: radius

Parabola $\begin{cases} x = 2at \\ y = at^2 \end{cases}$

- a: focal length

8.1.2 Other parameterisations

- t for time in 2 dimensional motion, such as projectile motion.

$$\begin{cases} x = Vt\cos\alpha \\ y = -\frac{1}{2}gt^2 + Vt\sin\alpha \end{cases}$$

- Previously in the legacy '4 Unit' course:

Ellipse $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$ $(a \neq b,\ a, b \neq 0)$

- a: semi major axis
- b: semi minor axis

Hyperbola $\begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases}$

- a: semi major axis

Rectangular hyperbola $\begin{cases} x = ct \\ y = \dfrac{c}{t} \end{cases}$

8.2 Conversion from Parametric to Cartesian form

- Remove parameter by solving simultaneously

Example 46

Find the Cartesian equation of $\begin{cases} x = 3t + 1 \\ y = 2t - 3 \end{cases}$.

Answer: $2x - 3y - 11 = 0$

Example 47

Find the Cartesian equation of $\begin{cases} x = 2q \\ y = q^2 - 3 \end{cases}$.

Answer: $x^2 = 4y + 12$

Additional questions Source: Fitzpatrick (1984, Ex 24(b))
Find the Cartesian equation of the curves whose parametric equations are:

1. $\begin{cases} x = 2t \\ y = t + 2 \end{cases} \quad (t \in \mathbb{R})$
2. $\begin{cases} x = t \\ y = t^2 \end{cases} \quad (t \in \mathbb{R})$
3. $\begin{cases} x = t \\ y = \frac{1}{t} \end{cases} \quad (t \in \mathbb{R})$
4. $\begin{cases} x = t + 3 \\ y = t^2 - 5 \end{cases} \quad (t \geq 0)$
5. $\begin{cases} x = 2u - 2 \\ y = 3u + 1 \end{cases} \quad (u \in [1, 3])$
6. $\begin{cases} x = v^3 \\ y = 1 - v^2 \end{cases} \quad (v \in [-1, 1])$
7. $\begin{cases} x = t + 2 \\ y = t^2 - 1 \end{cases} \quad (t \in \mathbb{R})$
8. $\begin{cases} x = 2t^2 \\ y = 4t \end{cases} \quad (t \in \mathbb{R})$
9. $\begin{cases} x = \dfrac{2t}{1+t^2} \\ y = \dfrac{1-t^2}{1+t^2} \end{cases} \quad (t \in \mathbb{R})$

Answers

1. $2y = x + 4$, $x \in \mathbb{R}$ **2.** $y = x^2$, $x \in \mathbb{R}$ **3.** $y = \frac{1}{x}$, $x \neq 0$ **4.** $x^2 - 6x + 4$, $x \in [3, \infty)$ **5.** $2y = 3x + 8$, $x \in [0, 4]$ **6.** $y = 1 - x^{\frac{2}{3}}$, $x \in [-1, 1]$ **7.** $y = x^2 - 4x + 3$, $x \in \mathbb{R}$ **8.** $y^2 = 8x$, $x \in [0, \infty)$ **9.** $x^2 + y^2 = 1$, $y \neq -1$

Further exercises

Ex 5H

- Q6-12

8.3 Other harder examples

Important note

Check the domain and/or range, or any other asymptotes that may or may not appear.

Example 48

[Ex 9D Q5] Find the Cartesian equation: $\begin{cases} x = t + \dfrac{1}{t} \\ y = t^2 + \dfrac{1}{t^2} \end{cases}$

Answer: $y = x^2 - 2$, $x \leq -2$ or $x \geq 2$

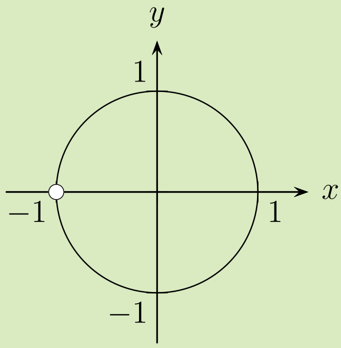
Example 49

[2025 CSSA Ext 1 Trial Q8] A curve is defined parametrically as

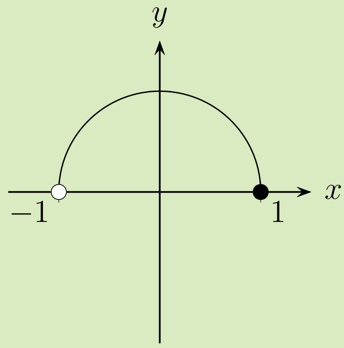
$$\begin{cases} x = \dfrac{1-t^2}{1+t^2} \\ y = \dfrac{2t}{1+t^2} \end{cases} \qquad \text{for } t \in [0, \infty)$$

Which of the following graphs best represents the curve? **Answer:** (C)

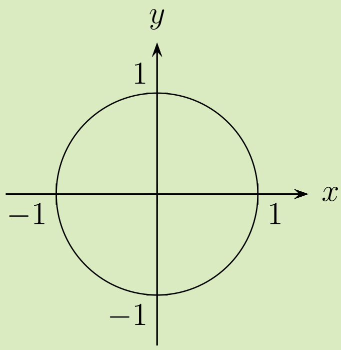
(A)



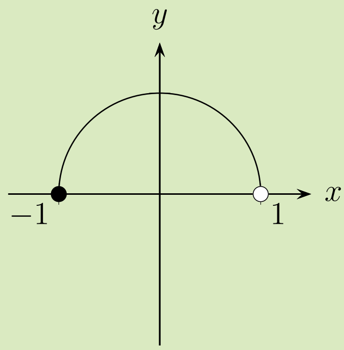
(C)



(B)



(D)



References

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Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019). *CambridgeMATHS Stage 6 Mathematics Extension 1 Year 11* (1st ed.). Cambridge Education.