



NORMANHURST BOYS HIGH SCHOOL

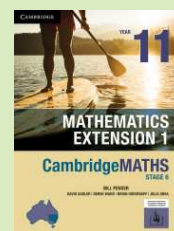
MATHEMATICS EXTENSION 1 YEAR 11 COURSE



Topic summary and exercises:

(X1) Further work with functions

With references to



Name:

Initial version by H. Lam, 2019. Last updated January 18, 2026.

Based on the work from the legacy syllabuses by R. Trenwith, 1995–2010, subsequently maintained by H. Lam, 2011–2026.

Various corrections by students & members of the Department of Mathematics at North Sydney Boys and Normanhurst Boys High Schools.

The questions and structure of document of the *Graphical Techniques* section arise from various sources, including

- *Graphs* Books 1 & 2, P. Marsh.
- *Sketching Graphs, Extension 2*, R. Trenwith.
- Assistance from M. Ziariaris.
-  GeoGebra animations by E. Chiem (Baulkham Hills High School)

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at <http://www.flaticon.com>, used under  CC BY 2.0.

Symbols used

-  Beware! Heed warning.
-  Mathematics Advanced content, for MAA coded class
-  Mathematics Extension 1 content, for MAX coded class
-  Literacy: note new word/phrase.
-  Facts/formulae to memorise.
-  On the course Reference Sheet.
- $\mathbb{N}$ the set of natural numbers
- $\mathbb{Z}$ the set of integers
- $\mathbb{Q}$ the set of rational numbers
- $\mathbb{R}$ the set of real numbers
- $\forall$ for all

Content outcomes addressed

ME1-11-01 Solves problems involving inequalities, functions and their inverses, graphical relationships between functions, and parametric equations

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Further exercises from
 - *Cambridge Year 11 Extension 1*
(Pender, Sadler, Ward, Dorofaeff, & Shea, 2019)will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Learning intentions & outcomes

✓ Content

☐ 3.1 Inequalities

- Solve cubic inequalities where the cubic is expressed as a product of linear factors.
-  Solve inequalities involving rational expressions with variables in the denominator.
- Solve absolute value inequalities of the form $|ax + b| \geq k$, $|ax + b| \leq k$, $|ax + b| < k$, $|ax + b| > k$, where a , b and k are constants, using algebraic and graphical methods or the characterisation of $|x|$ as the distance of x from the origin.

☐ 3.2 Graphical relationships

- Examine the relationship between the graph of $y = f(x)$ and the graph of $y = \frac{1}{f(x)}$ using graphing applications, and graph $y = \frac{1}{f(x)}$ given $y = f(x)$ in algebraic or graphical form, identifying any vertical and horizontal asymptotes of $y = f(x)$ and $y = \frac{1}{f(x)}$.
 - Examine the relationship between the graphs of $y = f(x)$ and $y = g(x)$ and the graphs of $y = f(x) + g(x)$ and $y = f(x) - g(x)$ using graphing applications, and graph $y = f(x) + g(x)$ and $y = f(x) - g(x)$ given $y = f(x)$ and $y = g(x)$ in algebraic or graphical form.
 - Determine the domains and ranges of the sum and difference of functions where possible, and, if appropriate, verify them using a graphing application.
 - Examine the relationship between the graph of $y = f(x)$ and the graphs of $y = |f(x)|$ and $y = f(|x|)$ using graphing applications, and graph $y = |f(x)|$ and $y = f(|x|)$ given $y = f(x)$ in algebraic or graphical form.
 - Apply knowledge of graphical relationships to solve problems involving graphs of functions, justifying conclusions in the context of the problem where appropriate.
-

☐ 3.3 Inverse functions

- Describe a function as one-to-one if every element in the range of the function corresponds to exactly one element of the domain.
 - Establish that the reflection of a point in the line $y = x$ reverses the coordinates of the point.
 - Define an inverse function f^{-1} informally as a function that reverses or undoes the effect of the function f .
 - Recognise that inverse functions exist for one-to-one functions.
 - Determine the equation for the inverse function $f^{-1}(x)$ of a given one-to-one function $y = f(x)$ by interchanging the variables x and y in $y = f(x)$.
 - Compare the graphs of a function and its inverse function using graphing applications, and recognise that the two graphs are reflections of each other in the line $y = x$.
 - Establish that the domain of $f^{-1}(x)$ is the range of $f(x)$, and the range of $f^{-1}(x)$ is the domain of $f(x)$, and use this relationship to solve problems.
 - Graph the inverse function of a given one-to-one function.
 - Explain that the reflection in the line $y = x$ exchanges horizontal and vertical lines and recognise that the horizontal line test can therefore be applied to the graph of $y = f(x)$ to determine whether its reflection in the line $y = x$ is a function.
 - Apply restrictions to the domain of a function, if it is not one-to-one, to obtain an inverse function.
 - Define formally $f^{-1}(x)$ to be the inverse function of $f(x)$ if the relationships $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ hold, and use this definition to solve problems.
 - Solve problems based on the relationship between a function and its inverse function using algebraic and graphical techniques, including determining the points of intersection of a function and its inverse, where they exist.
-

✓ Content

☐ 3.4 Parametric form of a function or relation

- Recognise that a curve may be represented by two parametric equations that give x and y as functions of a parameter and that the curve may also have a Cartesian equation in x and y .
 -  Express linear and quadratic functions and circles in parametric form.
 - Convert linear and quadratic functions and circles from parametric form to Cartesian form.
 - Graph linear and quadratic functions and circles expressed in parametric form.
-

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Part I

Inequalities

Section 1

≈ Linear inequalities

Important note

- The sign of the inequality will when or by a number.
- Split any double-ended inequalities into two separate ones.



Example 1

Solve $20 > 2 - 3x \geq 8$, sketching the solution on a number line. **Answer:** $x \in (-6, -2]$



Further exercises

Ex 5A

- Q2-3

Section 2

Quadratic, cubic and rational inequalities

2.1 Quadratic



Learning Goal(s)



Knowledge

Solving cubic inequalities



Skills

Determine the range of values satisfying an inequality



Understanding

The need to examine the graph to efficiently solve inequalities

✓ By the end of this section am I able to:

3.1 Inequalities 

- Solve cubic inequalities where the cubic is expressed as a product of linear factors.



Important note

-  Draw picture! Solving it purely algebraically will definitely cause problems.
- A polynomial inequality should be solved by the polynomial and solved



Example 2

Solve:

(a) $x^2 > 9$

(b) $x + 6 \geq x^2$

Answer: (a) $x \in (-3, 3)$ (b) $x \in [-2, 3]$

2.2 Cubic inequalities



Example 3

Sketch the graph of $y = (x + 2)(x + 1)(x - 3)$ and hence find the values of x where

$$(x + 2)(x + 1)(x - 3) < 0$$



Further exercises

Ex 5A

- Q4-6, 9, 13, 16

Ex 5B

- Q1, 4-5, 8

Additional questions

1. Determine the zeros and sketch their graphs, clearly indicating where they cut the x axis:

(a) $y = x(x+2)(x+1)(x-3)$ (c) $y = x(x+1)(x-2)^2$

(b) $y = (x+3)(x+1)(x-1)(3-x)$ (d) $y = (x+1)(x-2)^3$

2. Solve using graphical means:

(a) $(x-2)(x+3) < 0$ (d) $x^2 + 5x + 6 > 0$ (g) $x^3 - 5x^2 + 4x \geq 0$

(b) $(x-2)(x+3) > 0$ (e) $x(x+1)(x-1) \leq 0$ (h) $x(x-2)^2 > 0$

(c) $x^2 + 5x + 6 \leq 0$ (f) $-x(x+1)(x-1) \leq 0$ (i) $x(x-2)^2 \leq 0$

Answers

1. Check via  GeoGebra or Desmos 2. (a) $-3 < x < 2$ (b) $x < -3, x > 2$ (c) $-3 \leq x \leq -2$ (d) $x < -3, x > -2$ (e) $x \leq -1, 0 \leq x \leq 1$ (f) $-1 \leq x \leq 0, x \geq 1$ (g) $0 \leq x \leq 1, x \geq 4$ (h) $x > 0, x \neq 2$ (i) $x \leq 0, x = 2$

2.3 Rational inequalities



Learning Goal(s)

Knowledge

Solving rational expression inequalities

Skills

Identifying when to multiply through by the square of the denominator in inequalities

Understanding

The uncertainty involved in inequalities with unknowns in the denominator

By the end of this section am I able to:

3.1 Inequalities 



Solve inequalities involving rational expressions with variables in the denominator.



Fill in the spaces

An inequality of the form $y = \frac{f(x)}{g(x)}$ with an unknown in the denominator will make it uncertain where the function becomes positive or negative.

To avoid this problem,

- multiply by the of the denominator, or,
- and solve graphically.

2.3.1 Expressions resulting in a quadratic inequality

Gentle reminder

What is your favourite f -word? 😊

.....

Example 4

Solve $\frac{5}{x-4} \geq 1$

Graphical

Algebraic

Important note

 Remember to exclude a particular value of x !

**Example 5**

For what values of x is $\frac{2x+3}{x-4} > 1$?

Answer: $x < -7$ or $x > 4$.

Graphical

Algebraic



Important note



Remember to exclude a particular value of x !

2.3.2 Expressions resulting in a polynomial inequality

**Example 6**

Solve $\frac{6}{x} \geq x - 5$.

Answer: $0 < x \leq 6$ or $x \leq -1$.

Graphical

Algebraic



Important note



Remember to exclude a particular value of x !

**Example 7**

[2002 NEAP Ext 1 Q3] (3 marks) Solve the inequality $\frac{x}{x^2 - 4} \leq 0$ for x .

Answer: $x \in \{(-\infty, -2) \cup [0, 2)\}$.

**Important note**

⚠ Remember to exclude particular value(s) of x !

Additional questions

Source Portions taken from [Fitzpatrick \(1984, Ex 23\(a\)\)](#)

- | | | |
|-----------------------------|------------------------------------|---|
| 1. $x^2 - 2x - 15 \leq 0$ | 13. $\frac{4x-3}{2x+1} \leq 3$ | 22. $\frac{12}{3x+2} > 4$ |
| 2. $x(x-1) \leq 6$ | 14. $\frac{1}{2x-1} \leq 2$ | 23. $\frac{6}{5x-2} < 2$ |
| 3. $4x^2 - 12x + 10 > 0$ | 15. $\frac{2}{1-x} > -1$ | 24. $\frac{1}{(x-1)(x-3)} \leq -1$ |
| 4. $x^2 + 4x + 13 \leq 0$ | 16. $\frac{1}{x-2} > 1$ | 25. $\frac{7}{(3-x)(x+3)} > -1$ |
| 5. $-3x^2 + 10x + 8 \leq 0$ | 17. $\frac{2}{2-x} \geq 3$ | 26. $3x^2 + 5x + 1 > \frac{1}{4}$ |
| 6. $2x^2 + 5x + 2 \geq 0$ | 18. $\frac{1}{x} < \frac{1}{4}$ | 27. $\frac{2x-4}{x+3} > \frac{x+2}{2x+6}$ |
| 7. $(x-1)(x+3)(x-2) < 0$ | 19. $\frac{1}{x-3} > 2$ | 28. $\frac{1}{5x^2-2x-7} < \frac{5}{13}$ |
| 8. $(2+x)(x-5)(x+1) > 0$ | 20. $\frac{8}{x+1} > 2$ | 29. $2^{2x} - 5(2^x) + 4 \leq 0$ |
| 9. $x^2(x-1) \leq 0$ | 21. $\frac{4}{2x-1} < \frac{2}{3}$ | 30. $2^{2x} - 2(2^x) \leq -1$ |
| 10. $\frac{x-3}{x+1} > 0$ | | |
| 11. $\frac{1}{x} > 6$ | | |
| 12. $\frac{x-2}{x+3} > -2$ | | |

31.  Find values of x for which the following inequations are simultaneously satisfied:

- (a) $\frac{x+4}{x+6} < 0$ and $\frac{x-6}{x-4} > 1$
- (b) $x + \frac{1}{|x|} > 0$ and $x^2 - x - 2 < 0$
- (c) $x^2 - 5x + 4 \leq 0$ and $6 - x - x^2 > 0$

Answers

1. $-3 \leq x \leq 5$ 2. $-2 \leq x \leq 3$ 3. $x \in \mathbb{R}$ 4. no solution 5. $x \leq -\frac{2}{3}$ or $x \geq 4$ 6. $x \leq -2$ or $x \geq -\frac{1}{2}$ 7. $x < -3$ or $1 < x < 2$
 8. $-2 < x < -1$ or $x > 5$ 9. $x \leq 1$ 10. $x < -1$ or $x > 3$ 11. $0 < x < \frac{1}{6}$ 12. $x < -3$ or $x > -\frac{4}{3}$ 13. $x \leq -3$ or $x > -\frac{1}{2}$
 14. $x < \frac{1}{2}$ or $x \geq -\frac{4}{3}$ 15. $x < 1$ or $x > 3$ 16. $2 < x < 3$ 17. $\frac{4}{3} \leq x < 2$ 18. $x < 0$ or $x > 4$ 19. $3 < x < \frac{7}{2}$ 20. $-1 < x < 3$
 21. $x < \frac{1}{2}$ or $x > \frac{7}{2}$ 22. $-\frac{2}{3} < x < \frac{1}{3}$ 23. $x < \frac{2}{5}$ or $x > 1$ 24. $1 < x < 3$ 25. $x < -4$ or $-3 < x < 3$ or $x > 4$
 26. $x > -\frac{1}{6}$ or $x < -\frac{3}{2}$ 27. $x < -3$ or $x > \frac{10}{3}$ 28. $x < -\frac{6}{5}$ or $-1 < x < \frac{7}{5}$ or $x > \frac{8}{5}$ 29. $0 \leq x \leq 2$ 30. $x = 0$
 31. (a) $-6 < x < -4$ (b) $-1 \leq x < 2$ (c) $-1 < x < 0$ or $0 < x < 2$

Further exercises

Ex 5A

- Q11, 14

Ex 5B

- Q7, 10

Section 3

Absolute value inequalities



Learning Goal(s)



Knowledge

Solving absolute value inequalities



Skills

Identifying when to multiply through by the square of the denominator in inequalities



Understanding

The uncertainty involved in inequalities with unknowns in the denominator

✓ By the end of this section am I able to:

3.1 Inequalities

- Solve absolute value inequalities of the form $|ax + b| \geq k$, $|ax + b| \leq k$, $|ax + b| < k$, $|ax + b| > k$, where a , b and k are constants, using algebraic and graphical methods or the characterisation of $|x|$ as the distance of x from the origin.

3.1 Harder absolute value equations



Important note



Draw picture! Solving it purely algebraically may cause problems.

3.1.1 Absolute value with additional variables



Example 8



Solve $|2x - 1| = 3$.

Answer: $x = -1, 2$

**Example 9**

Solve $|3x - 7| = |2x - 3|$.

**Important note**

Draw picture to verify the number of solutions.

**Example 10**

Find t if $t + |t| = 8$

**Example 11**

Find a if $|a - 3| + a + 3 = 0$

3.1.2 Absolute value by cases



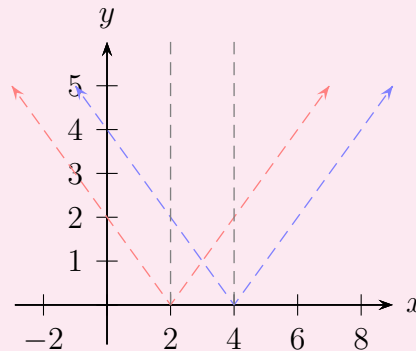
Example 12

- (a) Sketch $y = |x - 2| + |x - 4|$.
- (b) Hence or otherwise, solve $|x - 2| + |x - 4| = 4$.



Steps

1. Sketch $y = |x - 2|$ and $y = |x - 4|$ on the same set of axes:



2. Perform addition of ordinates and take branches to sketch $y = |x - 2| + |x - 4|$:

- $x < 2$
- $2 < x < 4$
- $x > 4$

(Draw on the same set of axes as previous part)

3. Examine the three 'branches' of $y = |x - 2| + |x - 4|$:

- | | |
|---|---|
| <ul style="list-style-type: none"> • $x < 2$: ➤ branch $y = x - 2$ ➤ branch $y = x - 4$ $\therefore y = \dots\dots\dots$ | <ul style="list-style-type: none"> • $x > 4$: ➤ branch $y = x - 2$ ➤ branch $y = x - 4$ $\therefore y = \dots\dots\dots$ |
|---|---|

- $2 < x < 4$:
- branch $y = |x - 2|$
- branch $y = |x - 4|$
- $\therefore y = \dots$

4. Solve $\begin{cases} y = |x - 2| + |x - 4| \\ y = 4 \end{cases}$ simultaneously by sketching $y = 4$, and examining intersection with appropriate branches.

**Example 13**

- (a) Sketch $y = |x + 1| + |x - 2|$.
- (b) Hence or otherwise, solve $|x + 1| + |x - 2| = 6$.

**Example 14**

Solve $|x - 3| + |x + 1| = 4$.

**Important note**

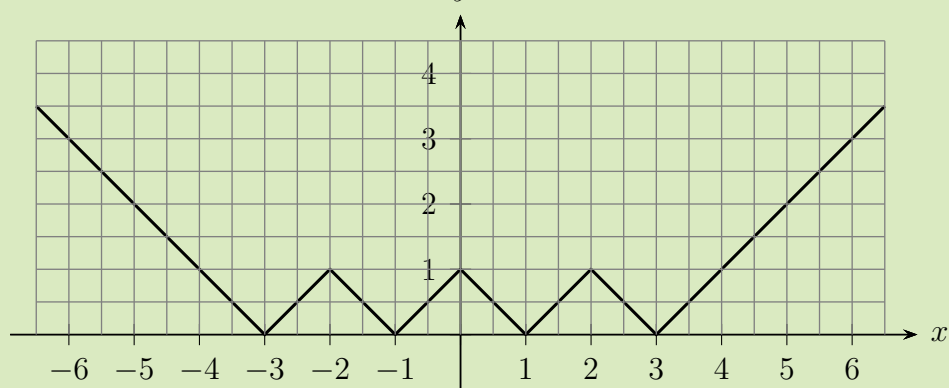
- What does Example 14 highlight about making an accurate sketch and understanding the graphs?

3.1.3 Nasty looking questions



Example 15

[2023 Ext 1 HSC Q8] The diagram shows the graph of a function.



Which of the following is the equation of the function?

Answer: (A)

(A) $y = |1 - ||x| - 2||$

(C) $y = |1 - |x - 2||$

(B) $y = |2 - ||x| - 1||$

(D) $y = |2 - |x - 1||$



Important note

The most efficient problem solving technique in this instance:

3.2 Solution of inequalities



Example 16

[2012 2U Q11] (2 marks) Solve $|3x - 1| < 2$.



Important note

Which two graphs should be sketched and their points of intersection found in order to verify the solution?



Example 17

(a) Solve $|2x - 5| = x + 2$.

Answer: $x = 1, 7$

(b) Hence solve $|2x - 5| \geq x + 2$.

**Example 18****[2016 NBHS Ext 1 Assessment Task 1] ⚠**

- (i) On the same set of axes, sketch the graphs of $y = 2|x|$ and $y = |x + 3|$. **4**
- (ii) Give the coordinates of the point(s) of intersection of the two graphs. **2**
- (iii) Hence solve $2|x| > |x + 3|$. **1**

**Example 19**

A Sketch $y = \frac{|x - 1|}{(x - 1)(x + 3)}$.

**Example 20**

[2013 Ext 1 Q10]  Which inequality has the same solutions as

$$|x + 2| + |x - 3| = 5$$

(A) $\frac{5}{3-x} \geq 1$

(C) $x^2 - x - 6 \leq 0$

(B) $\frac{1}{x-3} - \frac{1}{x+2} \leq 0$

(D) $|2x - 1| \geq 5$

**Example 21**

[2025 Independent Ext 1 Trial Q11] Solve for x :

$$|x - 5| \leq 3$$

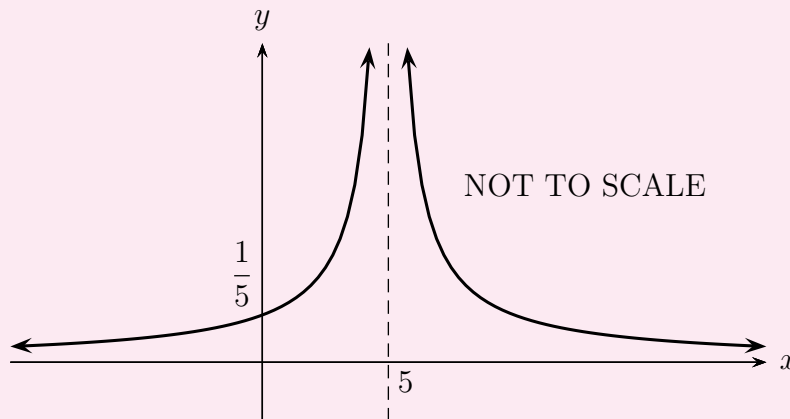
Answer: $x \in [-8, -2] \cup [2, 8]$

**Important note**

$|x| < k$ can also be expressed as $< x < \dots$.

⌚ **Timed exam practice 1 (Allow approximately 5 minutes)**

[2024 Ext 1 HSC Q12] (3 marks) The diagram shows the graph of $y = \frac{1}{|x - 5|}$.



For what values of x is $\frac{x}{6} \geq \frac{1}{|x - 5|}$?

Marking criteria

- ✓ [1] Recognises the two cases *or* observes that $x = 6$ is a critical point *or* graphs the two curves on the same axis, or equivalent merit.
- ✓ [2] Obtains the three critical points 2, 3 and 6, or equivalent merit.
- ✓ [3] Provides correct solution.

Additional questions

1. Find all solutions to the following:

- | | | |
|---------------------------|----------------------------|---------------------------|
| (a) $ 3x - 7 = 2x - 3 $ | (e) $ x + 1 = 2x + 7 $ | (i) $ 2x + 1 = x - 2 $ |
| (b) $ 7 - 2x = x - 2 $ | (f) $ 5 - 3x = x + 3 $ | (j) $2 x + 8 = 3 x + 5 $ |
| (c) $ 2 + x = -x $ | (g) $ 7x - 4 = 3x + 16 $ | (k) $5 x - 7 = 9x + 1 $ |
| (d) $ 3x - 1 = 5 + 2x $ | (h) $ 9x + 2 = 3x - 4 $ | (l) $ 7x - 3 = 4 x + 6 $ |

2. Solve each of the following and verify that the solutions are valid:

- | | | |
|--------------------------|-------------------------|--------------------------|
| (a) $ 2x - 1 = x + 7$ | (e) $ 5 - 3x = x + 1$ | (i) $ 2x = 9 - x$ |
| (b) $ x + 7 = 2x - 1$ | (f) $5 - 3x = x + 1 $ | (j) $ 2x + 5 = 3x + 9$ |
| (c) $ 2x - 11 = 3x - 4$ | (g) $ 3x + 1 = 2x + 4$ | (k) $ 6x - 5 = 5x + 27$ |
| (d) $2x - 11 = 3x - 4 $ | (h) $ 4x - 1 = 2x + 7$ | (l) $ 4 - 2x = x - 2$ |

3. Solve the following inequalities and graph each solution on the number line.

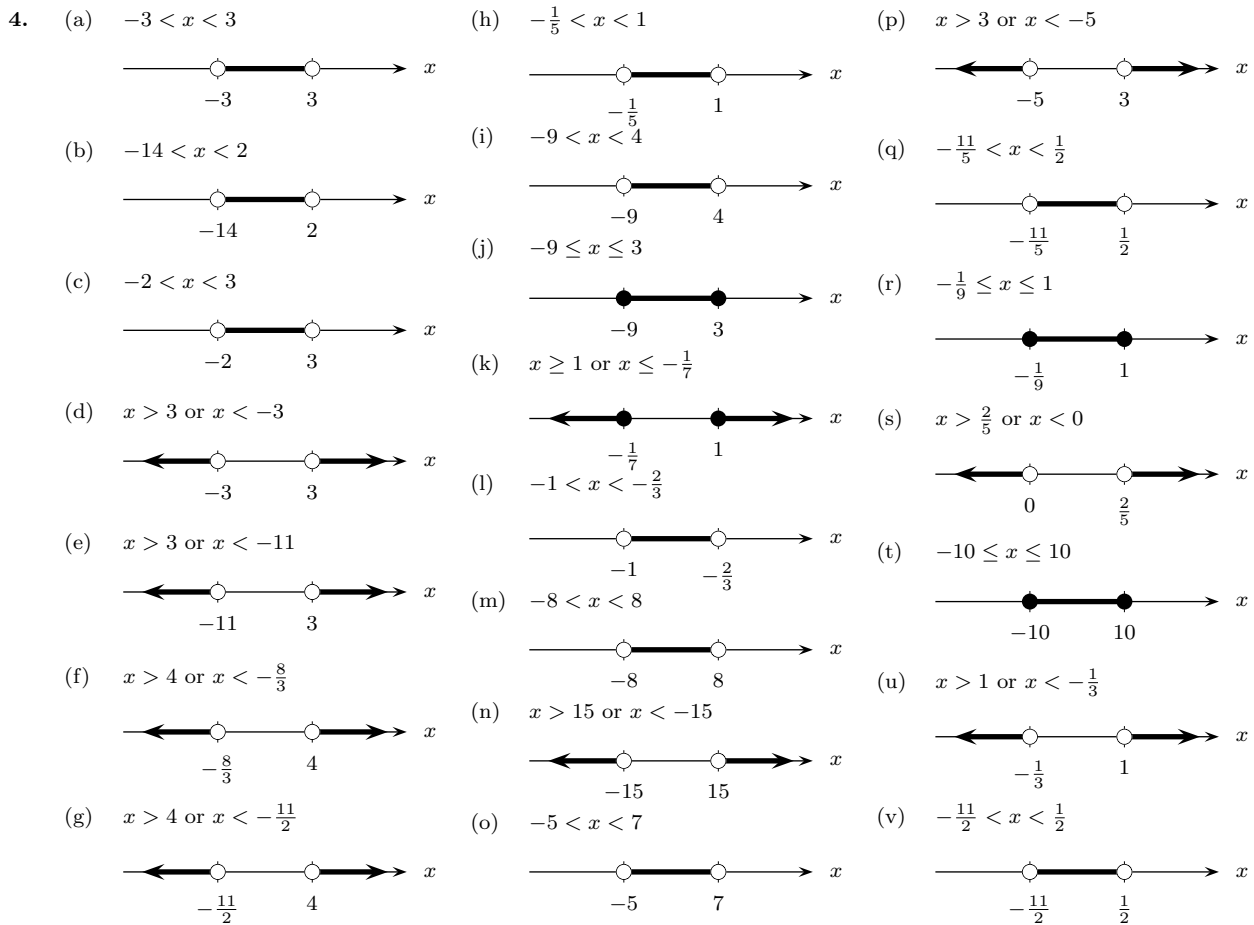
- | | | | |
|---------------------|------------------------------------|--|---|
| (a) $ 8x < 24$ | (h) $ 5x - 2 < 3$ | (n) $\left \frac{x}{3}\right > 5$ | (s) $ 5x - 1 > 1$ |
| (b) $ x + 6 < 8$ | (i) $ 2x + 5 < 13$ | (o) $\left \frac{x - 1}{2}\right < 3$ | (t) $\left \frac{2x}{5}\right \leq 4$ |
| (c) $ 2x - 1 < 5$ | (j) $ 2x + 6 \leq 12$ | (p) $\left \frac{x + 1}{4}\right > 1$ | (u) $\left \frac{3x - 1}{2}\right > 1$ |
| (d) $ x > 3$ | (k) $ 7x - 3 \geq 4$ | (q) $ 2x + 5 < 6$ | (v) $\left \frac{2x + 5}{3}\right < 2$ |
| (e) $ x + 4 > 7$ | (l) $ 6x + 5 < 1$ | (r) $ 9x - 4 \leq 5$ | |
| (f) $ 3x - 2 > 10$ | (m) $\left \frac{x}{2}\right < 4$ | | |
| (g) $ 4x + 3 > 19$ | | | |

4.  Solve by drawing appropriate sketches and examining branches:

- | | |
|-----------------------------|-----------------------------|
| (a) $ x + x - 4 = 2$ | (c) $ x - 3 + x + 1 = 2$ |
| (b) $ x - 3 + x + 1 = 5$ | (d) $ x - 1 + x + 3 = 4$ |

Answers

1. (a) $x = 2, 4$ (b) $x = 3, 5$ (c) $x = -1$ (d) $x = 6, -\frac{4}{5}$ (e) $x = -6, -\frac{8}{3}$ (f) $x = 4, \frac{1}{2}$ (g) $x = 5, -\frac{6}{5}$ (h) $x = -1, \frac{1}{6}$ (i) $x = \frac{1}{3}, -3$ (j) $x = 1, -\frac{31}{5}$ (k) $x = -9, \frac{17}{7}$ (l) $x = 9, -\frac{21}{11}$ 2. (a) $x = 8, -2$ (b) $x = 8$ (c) $x = 3$ (d) no solutions (e) $x = 1, 3$ (f) $x = 1$ (g) $x = -1, 3$ (h) $x = -1, 4$ (i) $x = 3, -9$ (j) $x = -\frac{14}{5}$ (k) $x = -2, 32$ (l) $x = 2$



5. (a) No solution (b) $x = -\frac{3}{2}, \frac{7}{2}$ (c) No solution (d) $-3 \leq x \leq 1$

Further exercises

Ex 5A

- Q12, 15, 17, 18

Part II

Graphical relationships

Section 4

Reflection about coordinate axes



Learning Goal(s)



Knowledge

Curve sketching techniques



Skills

Applying a curve sketching menu to graph functions



Understanding

The reflection of the graph $y = f(x)$ for the graphs $y = |f(x)|$ and $y = f(|x|)$

✓ By the end of this section am I able to:

3.2 Graphical relationships

- Examine the relationship between the graph of $y = f(x)$ and the graphs of $y = |f(x)|$ and $y = f(|x|)$ using graphing applications, and graph $y = |f(x)|$ and $y = f(|x|)$ given $y = f(x)$ in algebraic or graphical form.

4.1 Basic curve sketching techniques



Steps

1. Determine the
2. Look for symmetry (..... , even or).
3. Locate x and y
4. Determine where the graph is and where it is
5. Consider and asymptotes. Also consider the behaviour as $x \rightarrow \pm\infty$.

4.2 Reflection properties

- $f(-x)$: reflection about axis.
- $-f(x)$: reflection about axis.
- $|f(x)|$: reflect about axis.

➤ Use basic definition of absolute value:

$$|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$$

- $f(|x|)$:
 - Draw sketch of $y = f(x)$ for $x \geq 0$.
 - Reflect about y axis.

➤ Use basic definition of absolute value:

$$f(|x|) = \begin{cases} f(x) & x \geq 0 \\ f(-x) & x < 0 \end{cases}$$

4.3 ↻ Odd/Even functions

Definition 1

- An *odd* function has about the
Algebraically,
.....
- An *even* function has about the y axis. Algebraically,
.....

**Example 22**

Sketch the graph of $y = \sqrt{-x}$.

**Example 23**

Sketch the graph of $y = |x^2 - 2x|$.

**Example 24**

For $y = f(x) = (x - 2)(x - 4)$, sketch:

1. $f(x)$

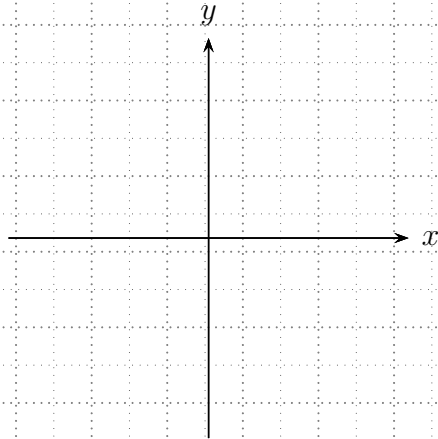
3. $-f(x)$

5. $f(|x|)$

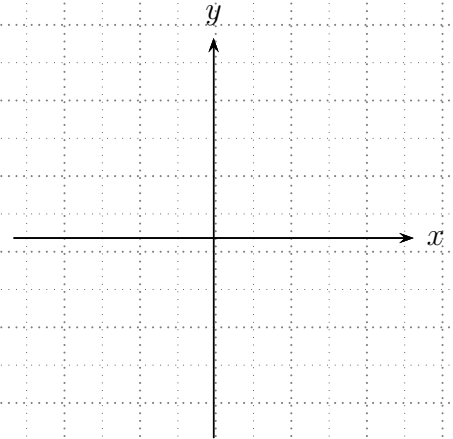
2. $f(-x)$

4. $|f(x)|$

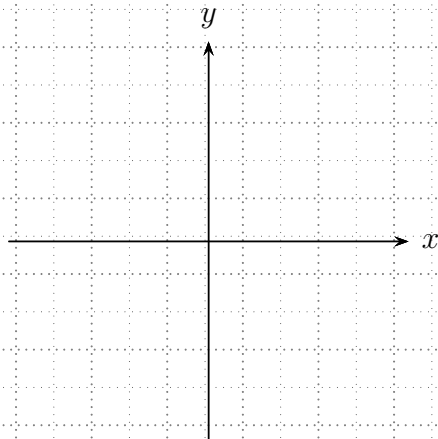
1.



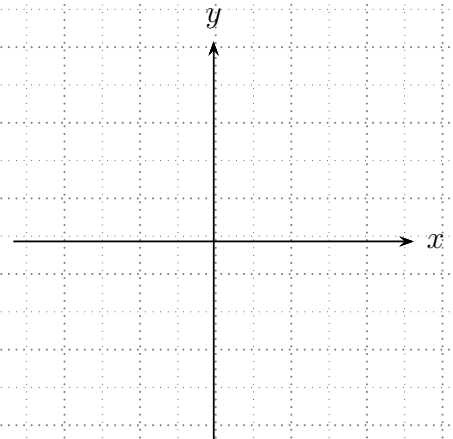
4.



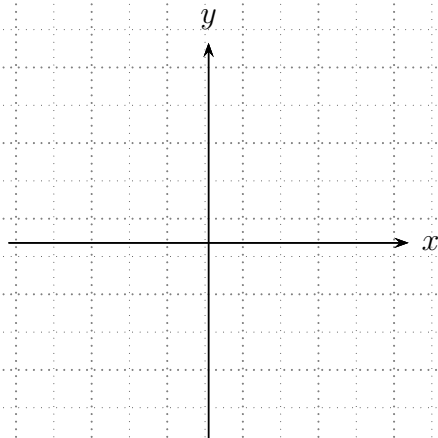
2.



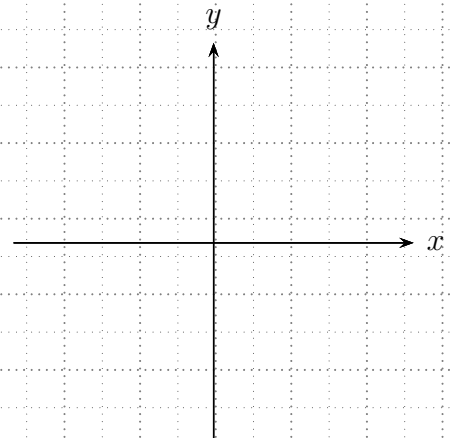
5.



3.



6.



Additional questions

Sketch the following graphs:

1. $y = \frac{1}{3^x} - 1$

2. $y = |x(x^2 - 4)|$

3. $y = 5|x| - 2$

4. $y = |x^2| - |x|$

5. $|y| = |x^2 - 1|$

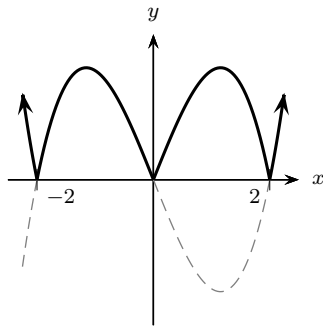
6. $|x| + |y| = 1$

7. $|x + y| = 1$

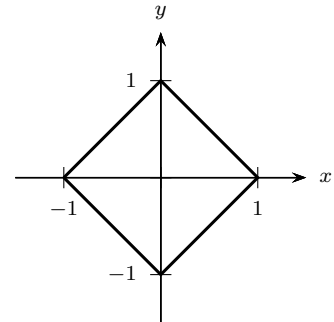
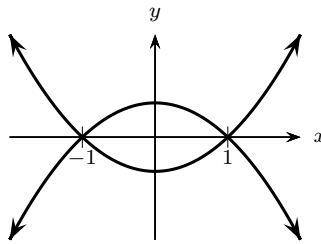
Answers

1. $y = \frac{1}{3^x} - 1$:  GeoGebra

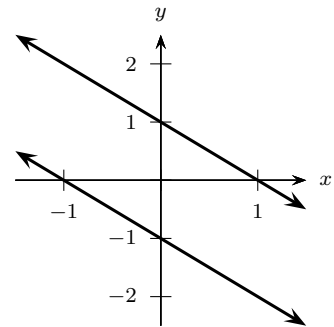
2. $y = |x(x^2 - 4)|$



5. $|y| = |x^2 - 1|$



7. $|x + y| = 1$



3. $y = 5|x| - 2$:  GeoGebra

4. $y = |x^2| - |x|$

• $x \geq 0$

$y = x^2 - x$

• $x < 0$,

$y = x^2 - (-x)$

6. $|x| + |y| = 1$
Consider cases:

• $y = 1 - |x|$

• $-y = 1 - |x|$

Also, $x, y < 1$ for this to be true. **Further exercises****Ex 4B**

- Q5-6

Ex 5E

- Q2, 5-6, 8

-
- 
- Q10, 14

Section 5

Addition/subtraction of ordinates: $y = f(x) \pm g(x)$



Learning Goal(s)

Knowledge

Curve sketching techniques

Skills

Applying a curve sketching menu to graph functions

Understanding

The addition/subtraction of ordinates is via the addition/subtraction of the y values of $f(x)$ and $g(x)$ at the same x value

 By the end of this section am I able to:

3.2 Graphical relationships 

- Examine the relationship between the graphs of $y = f(x)$ and $y = g(x)$ and the graphs of $y = f(x) + g(x)$ and $y = f(x) - g(x)$ using graphing applications, and graph $y = f(x) + g(x)$ and $y = f(x) - g(x)$ given $y = f(x)$ and $y = g(x)$ in algebraic or graphical form.
- Determine the domains and ranges of the sum and difference of functions where possible, and, if appropriate, verify them using a graphing application.



Steps

1. Ensure that the equation is written in the form $y = f(x) + g(x)$.
 - Test for even/odd function.
2. Find x and y intercepts for $y = f(x) + g(x)$.
3. Draw dotted sketches of $y = f(x)$ and $y = g(x)$ on the same set of axes.
4. Draw perpendicular lines to the x axis that cut $y = f(x)$ and $y = g(x)$.
5. Join points to represent graph of $y = f(x) + g(x)$.



GeoGebra



Adding-trig-curves.ggb



Example 25

Sketch $y = x + \frac{1}{x}$.

**Example 26**

Sketch $y = x - \frac{1}{x}$.

**Example 27**

Sketch $y = \sqrt{x-1} + \sqrt{5-x}$.

Additional questions

Sketch the following graphs.

1. $y = x + \frac{1}{x^2}$

2. $y = x - \frac{1}{x^2}$

3. $y = \frac{e^x + e^{-x}}{2}$ ($e \approx 2.71828 \dots$)

4. $y = \frac{e^x - e^{-x}}{2}$

5. $y = \frac{1}{x + e^x}$

6. $y = \frac{1}{x^2 - \frac{1}{x}}$

7. (a) Show that

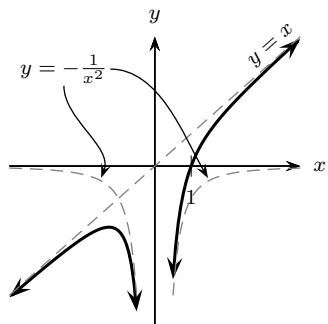
$$\frac{x^3 - x^2 + 1}{x - 1} = x^2 + \frac{1}{x - 1}$$

(b) Hence sketch $y = \frac{x^3 - x^2 + 1}{x - 1}$.

Answers

1. $y = x + \frac{1}{x^2}$:  GeoGebra

2. $y = x - \frac{1}{x^2}$

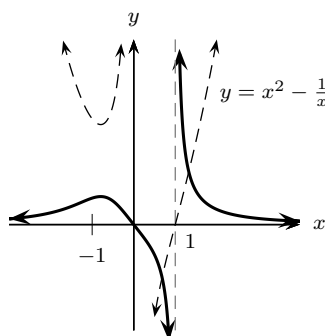


3. $y = \frac{e^x + e^{-x}}{2}$:  GeoGebra

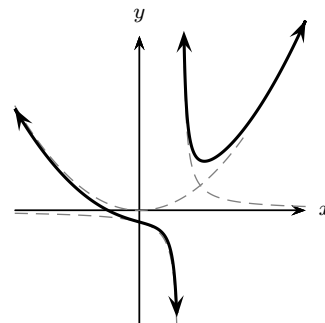
4. $y = \frac{e^x - e^{-x}}{2}$:  GeoGebra

5. $y = \frac{1}{x + e^x}$

6. $y = \frac{1}{x^2 - \frac{1}{x}}$



7. $y = \frac{x^3 - x^2 + 1}{x - 1}$



Section 6

Reciprocal graphs



Learning Goal(s)



Knowledge

Curve sketching techniques



Skills

Applying a curve sketching menu to graph functions



Understanding

$f(x) = 0$ generates vertical asymptotes, and $\lim_{x \rightarrow \pm\infty} \frac{1}{f(x)}$ gives the limiting behaviour.

✓ By the end of this section am I able to:

3.2 Graphical relationships ■

- Examine the relationship between the graph of $y = f(x)$ and the graph of $y = \frac{1}{f(x)}$ using graphing applications, and graph $y = \frac{1}{f(x)}$ given $y = f(x)$ in algebraic or graphical form, identifying any vertical and horizontal asymptotes of $y = f(x)$ and $y = \frac{1}{f(x)}$.



Steps

1. Draw $f(x)$ (draft)
2. Vertical asymptotes whenever $f(x) = 0$ as $y = \frac{1}{f(x)}$ does not exist.
3. Find the y intercept of $y = \frac{1}{f(x)}$.
4. Draw lines $y = \pm 1$, as $f(x)$ will pass through these exact same points.
5. Consider magnitude of $y = f(x)$ and hence magnitude of $y = \frac{1}{f(x)}$.
 - As $x \rightarrow \pm\infty$.
 - As x approaches asymptotes.



Important note

A maximum value on $f(x)$ will result in a value on $\frac{1}{f(x)}$, and vice versa.

⚠ Care is needed for graphs such as $y = \tan x$ and $y = \cot x$.

⚠ Later after the *Trigonometry* topic has been taught, when sketching $y = \frac{1}{\tan x}$ will take the shape of $y = \cot x$, except when $\tan x = 0$.



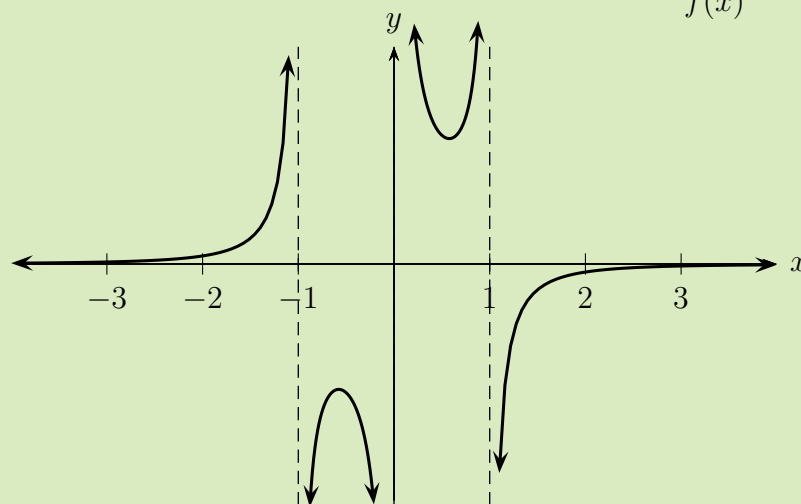
Example 28

Sketch $y = \frac{1}{(x-1)(x+1)(x-2)}$.



Example 29

[2022 NSGHS Ext 1 Trial Q7] The graph below shows $y = \frac{1}{f(x)}$.



Which of the following best represents the equation of $f(x)$?

Answer: (C)

(A) $f(x) = 1 - x^2$

(C) $f(x) = x(1 - x^2)$

(B) $f(x) = x(x^2 - 1)$

(D) $f(x) = x^2(x^2 - 1)$

6.1 Other manipulations involving $\frac{1}{f(x)}$

Laws/Results

Functions of the form $\frac{f(x)}{g(x)}$, $g(x) \neq 0$, follow the asymptotes.



Example 30

Sketch $y = \frac{x-1}{x-4}$, showing all important features.



Example 31

 Sketch $y = \frac{2x^2}{x^2-9}$, showing all important features.


Example 32

 Sketch $y = \frac{1}{x-2} + \frac{1}{x-8}$


Example 33

 Sketch $y = x^2 + \frac{1}{x^2}$.

**Example 34**

[Ex 3G Q17(a)]  Show that $\frac{(x-1)(x+2)}{x-3} = x + 4 + \frac{10}{x-3}$, and deduce that $y = \frac{(x-1)(x+2)}{x-3}$ has an oblique asymptote $y = x + 4$. Then sketch the graph.

**Example 35**

[Ex 3G Q17(b)]  Likewise, sketch $y = \frac{x^2 - 4}{x + 1} = x - 1 - \frac{3}{x + 1}$, showing the oblique asymptote.

Additional questions

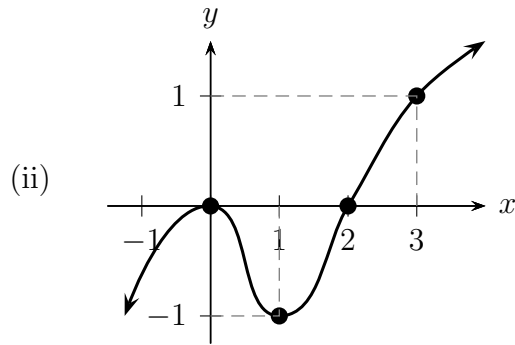
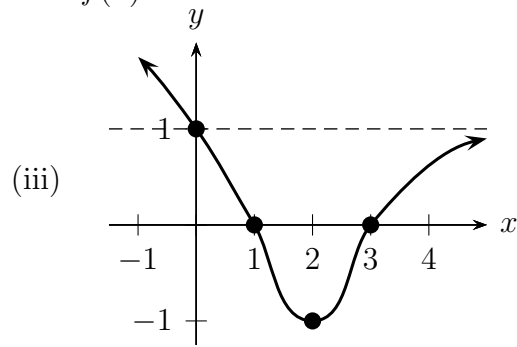
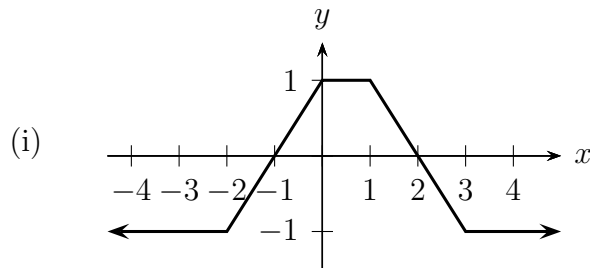
1. Sketch the following rectangular hyperbolae on separate number planes, stating the domain and range.

(a) $y = \frac{1}{x}$	(f) $y = \frac{2}{x-4} + 1$	(k) $y = \frac{2x+1}{x-2}$
(b) $y = \frac{3}{x-3}$	(g) $y = -2 - \frac{1}{x+1}$	(l) $y = \frac{3x-2}{x+3}$
(c) $y = \frac{-3}{x-3}$	(h) $y = \frac{1}{2x-3} - 3$	(m) $y = \frac{1-x}{1+x}$
(d) $y = 1 + \frac{1}{x+2}$	(i) $y = \frac{x+2}{x+1}$	
(e) $y = 4 - \frac{3}{x-3}$	(j) $y = \frac{x}{x+1}$	

2. Reciprocal functions

- (a) i. On the same set of axes, sketch the graphs of $y = x$ and $y = -\frac{1}{x}$.
- ii. By addition of ordinates, sketch the graph of $y = x - \frac{1}{x}$.
- (b) Sketch the following by the addition of ordinates:
- | | |
|----------------------------|------------------------------|
| i. $y = 2x + \frac{1}{x}$ | iii. $y = x^2 + \frac{1}{x}$ |
| ii. $y = \frac{3}{x} - 2x$ | iv. $y = x^2 - \frac{1}{x}$ |
- (c) Sketch the following graphs of reciprocal functions, stating the domain and range:
- | | |
|-----------------------------|----------------------------------|
| i. $y = \frac{1}{x^2}$ | vi. $y = \frac{1}{\sqrt{4-x^2}}$ |
| ii. $y = \frac{1}{(x-2)^2}$ | vii. $y = \frac{2}{(x-3)(x+2)}$ |
| iii. $y = \frac{1}{x^3-x}$ | viii. $y = \frac{1}{x^2+x}$ |
| iv. $y = \frac{2}{x^2+1}$ | ix. $y = \frac{x+1}{(x-1)(x+2)}$ |
| v. $y = \frac{2}{x^2-1}$ | x. $y = \frac{x^2-4}{x^2+2x-3}$ |

3. $f(x)$ is shown in the following diagrams. Sketch $\frac{1}{f(x)}$.



4. Make neat sketches of the following graphs:

(a) $y = 2 - x$ and $y = \frac{1}{2 - x}$

(g) $y = |x|$ and $y = \frac{1}{|x|}$

(b) $y = 3x + 1$ and $y = \frac{1}{3x + 1}$

(h) $y = 2^x$ and $y = \frac{1}{2^x}$

(c) $y = x^2 - 4$ and $y = \frac{1}{x^2 - 4}$

(i) $y = \sin x$ and $y = \frac{1}{\sin x}$

(d) $y = 9 - x^2$ and $y = \frac{1}{9 - x^2}$

(j) $y = (x - 1)(x^2 - 3x - 4)$
and $y = \frac{1}{(x - 1)(x^2 - 3x - 4)}$

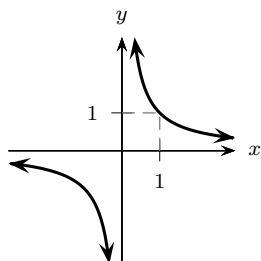
(e) $y = x^2 - x - 2$ and $y = \frac{1}{x^2 - x - 2}$

(f) $y = \sqrt{x}$ and $y = \frac{1}{\sqrt{x}}$

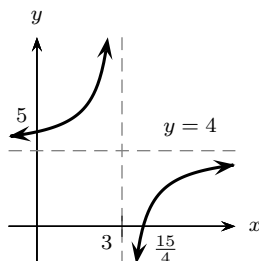
(k) $y = |2x + 1|$ and $y = \frac{1}{|2x + 1|}$

Answers

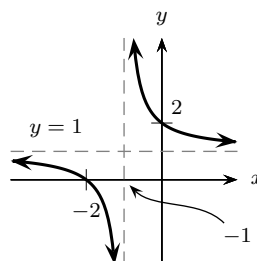
1. (a) $D = \{x : x \neq 0\}$
 $R = \{y : y \neq 0\}$



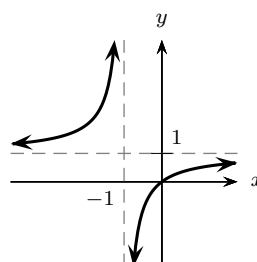
(e) $D = \{x : x \neq 3\}$
 $R = \{y : y \neq 4\}$



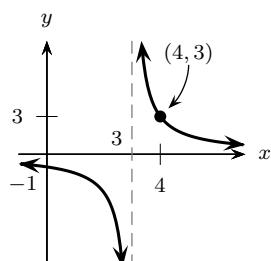
(i) $D = \{x : x \neq -1\}$
 $R = \{y : y \neq 1\}$



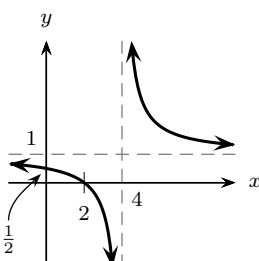
(j) $D = \{x : x \neq -1\}$
 $R = \{y : y \neq 1\}$



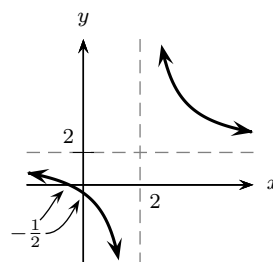
(b) $D = \{x : x \neq 3\}$
 $R = \{y : y \neq 0\}$



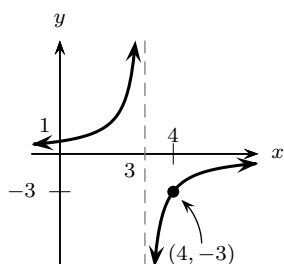
(f) $D = \{x : x \neq 4\}$
 $R = \{y : y \neq 1\}$



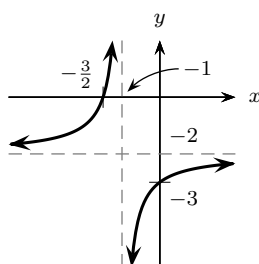
(k) $D = \{x : x \neq 2\}$
 $R = \{y : y \neq 2\}$



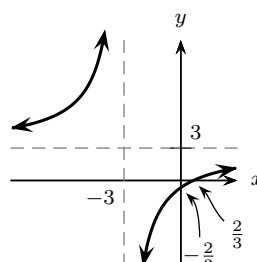
(c) $D = \{x : x \neq 3\}$
 $R = \{y : y \neq 0\}$



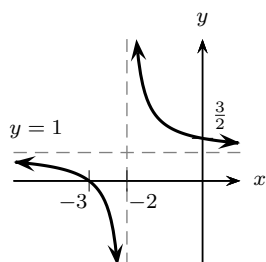
(g) $D = \{x : x \neq -1\}$
 $R = \{y : y \neq -2\}$



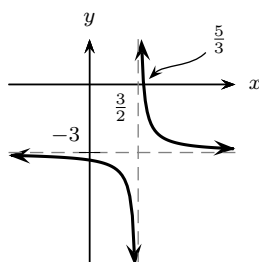
(l) $D = \{x : x \neq -3\}$
 $R = \{y : y \neq 3\}$



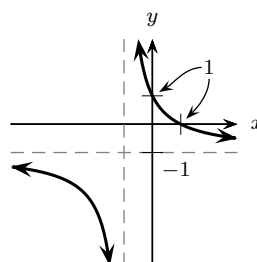
(d) $D = \{x : x \neq -2\}$
 $R = \{y : y \neq 1\}$



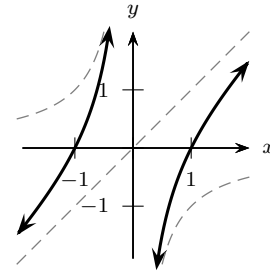
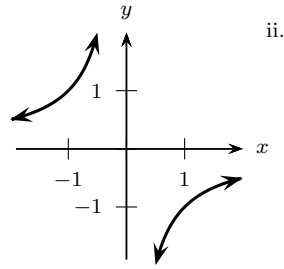
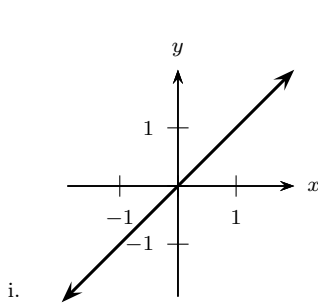
(h) $D = \{x : x \neq \frac{3}{2}\}$
 $R = \{y : y \neq -3\}$



(m) $D = \{x : x \neq -1\}$
 $R = \{y : y \neq -1\}$

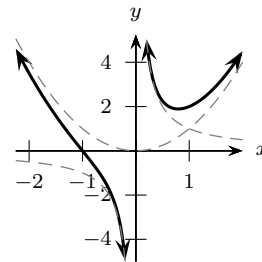
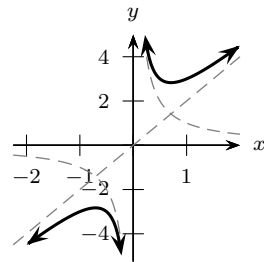


2. (a)



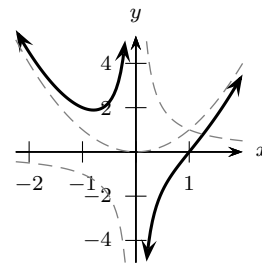
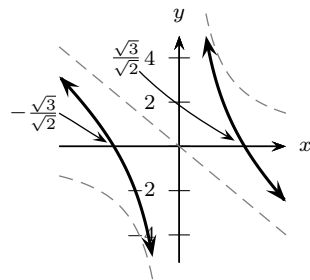
(b) i. $y = 2x + \frac{1}{x}$

iii. $y = x^2 + \frac{1}{x}$



ii. $y = \frac{3}{x} - 2x$

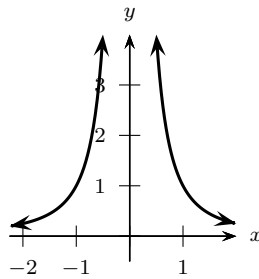
iv. $y = x^2 - \frac{1}{x}$



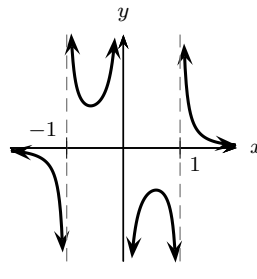
(c) i. $D = \{x : x \neq 0\}$
 $R = \{y : y > 0\}$

iii. $D = \{x : x \neq 0, \pm 1\}$
 $R = \{y : y \neq 0\}$

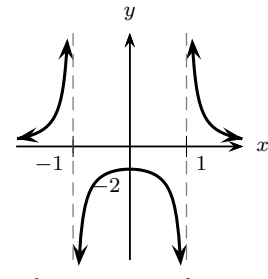
v. $D = \{x : x \neq \pm 1\}$
 $R = \{y : y \leq -2, y > 0\}$



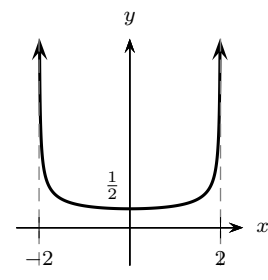
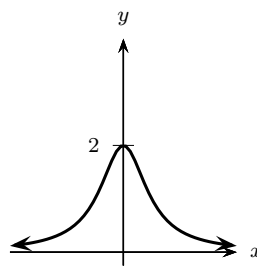
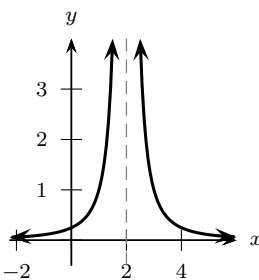
ii. $D = \{x : x \neq 2\}$
 $R = \{y : y > 0\}$



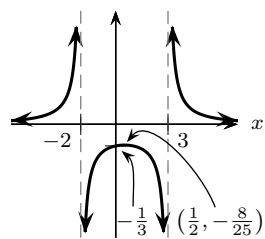
iv. $D = \{x : x \in \mathbb{R}\}$
 $R = \{y : y > 0\}$



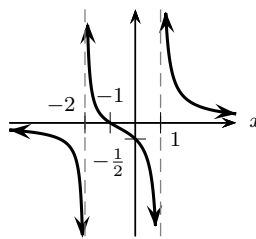
vi. $D = \{x : -2 < x < 2\}$
 $R = \{y : y > \frac{1}{2}\}$



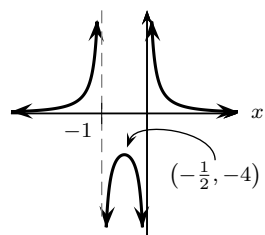
vii. $D = \{x : x \neq 3, -2\}$
 $R = \{y : y > 0, y \leq -\frac{8}{25}\}$



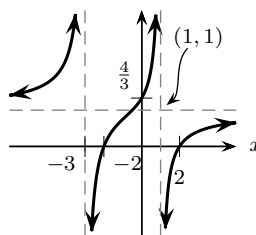
ix. $D = \{x : x \neq 1, -2\}$
 $R = \{y : y \in \mathbb{R}\}$



viii. $D = \{x : x \neq -1, 0\}$
 $R = \{y : y \leq -4, y > 0\}$



x. $D = \{x : x \neq -3, 1\}$
 $R = \{y : y \in \mathbb{R}\}$



3. Remaining exercises: check  GeoGebra where possible.

Further exercises

Ex 5C

- Q5-12, 14

Part III

Inverse functions and parametric representation

Section 7

Inverse functions



Learning Goal(s)

Knowledge

Properties of inverse functions

Skills

Determining if a function has an inverse function

Understanding

The difference between inverse relations and inverse functions

✓ By the end of this section am I able to:

3.3 Inverse functions

- Describe a function as one-to-one if every element in the range of the function corresponds to exactly one element of the domain.
- Establish that the reflection of a point in the line $y = x$ reverses the coordinates of the point.
- Define an inverse function f^{-1} informally as a function that reverses or undoes the effect of the function f .
- Recognise that inverse functions exist for one-to-one functions.
- Determine the equation for the inverse function $f^{-1}(x)$ of a given one-to-one function $y = f(x)$ by interchanging the variables x and y in $y = f(x)$.
- Compare the graphs of a function and its inverse function using graphing applications, and recognise that the two graphs are reflections of each other in the line $y = x$.
- Establish that the domain of $f^{-1}(x)$ is the range of $f(x)$, and the range of $f^{-1}(x)$ is the domain of $f(x)$, and use this relationship to solve problems.
- Graph the inverse function of a given one-to-one function.
- Explain that the reflection in the line $y = x$ exchanges horizontal and vertical lines and recognise that the horizontal line test can therefore be applied to the graph of $y = f(x)$ to determine whether its reflection in the line $y = x$ is a function.
- Apply restrictions to the domain of a function, if it is not one-to-one, to obtain an inverse function.
- Define formally $f^{-1}(x)$ to be the inverse function of $f(x)$ if the relationships $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ hold, and use this definition to solve problems.
- Solve problems based on the relationship between a function and its inverse function using algebraic and graphical techniques, including determining the points of intersection of a function and its inverse, where they exist.

7.1 Conditions for existence of an inverse function

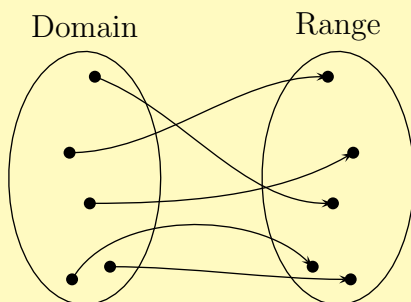
Definition 2



An *inverse relation* to $f(x)$ is denoted $f^{-1}(x)$. It “undo”s whatever $f(x)$ would do.

🔑 Laws/Results

A function only has an inverse if and only if it is



⚠️ Functions/relations will always have inverse *relations*.

⚠️ “Inverse” almost always implies “inverse, not inverse

📖 Definition 3

Monotonic functions a function is also *monotonic increasing* (or *monotonic decreasing*): if $b > a$, then $f(b) > f(a)$.

⚠️ Important note

⚠️ Asymptotes *may* cause functions to not be monotonic. Varies from case to case!

🎓 Example 36

Which of the following functions are monotonic?

1. $y = 2x$

2. $y = x^2$

3. $y = \frac{2x}{x+2}$

4. $y = x - \frac{1}{x}$

Additional questions

Source [Fitzpatrick \(1984, Ex 26\(a\)\)](#)

1. State whether the following are one-to-one functions.

(a) $f(x) = x - 2, x \in \mathbb{R}$

(e) $f(x) = \frac{1}{9-x}, x \neq 9$

(b) $f(x) = x^2 - 4x + 1, x \geq 2$

(f) $f(x) = |9 - x|, x \in \mathbb{R}$

(c) $f(x) = \sqrt{4 - x^2}, x \in [-2, 2]$

(g) $f(x) = 9 - x^2, x \in \mathbb{R}$

(d) $f(x) = 9 - x, x \in \mathbb{R}$

(h) $f(x) = x^3 - 4x, x \in [-2, 2]$

2. Find the largest possible domain for which the following are one-to-one (monotonic increasing) functions.

(a) $f(x) = \sqrt{4 - x^2}$

(d) $f(x) = 3x - x^2$

(b) $f(x) = \sqrt{x^2 - 4}$

(e) $f(x) = x^2 + 6x + 8$

(c) $f(x) = -\frac{1}{x+2}$

3. Explain why the following functions do not have an inverse function. Suggest suitable restrictions to their domain so that the restricted function may have an inverse.

(a) $f(x) = \sqrt{a^2 - x^2}, x \in [-a, a]$

(b) $f(x) = 4 - x^2$

(c) $f(x) = \frac{1}{x^2}, x \neq 0$

Answers

1. (a) Yes (b) Yes (c) No (d) Yes (e) Yes (f) No (g) No (h) No 2. (a) $x \in [-2, 0]$ (b) $x \geq 2$ (c) $x < -2$ (d) $x \leq \frac{3}{2}$ (e) $x \geq -3$

3. These are some of the possible domain restrictions. There are others which will also work. (a) $x \in [0, a]$ (b) $x \in [0, 2]$ (c) $x \in (0, \infty)$

7.3 Finding the inverse relation



Steps

1. Interchange x and y .
2. Change subject to y .



Example 37

Find the inverse function of $y = x + 1$.

**Example 38**

Find the equations of the inverse relations to these functions. If the inverse is a function, find an expression for $f^{-1}(x)$, and verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

(a) $f(x) = 6 - 2x, x > 0.$

(c) $f(x) = \frac{1-x}{1+x}.$

(b) $f(x) = x^3 + 2.$

(d) $f(x) = x^2 - 9.$

**Example 39**

- (a) Find the inverse function of $y = \frac{2}{x-1}$.
- (b) Sketch the inverse on the same number plane as the original function.

**Example 40**

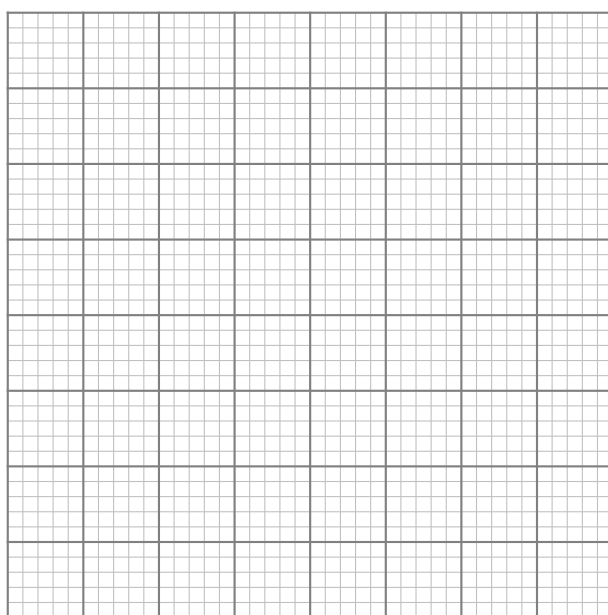
For the function of $f(x) = \frac{x+1}{x-2}$,

Answer: (a) $y = 1 + \frac{3}{x-2}$ (b) $y = 2 + \frac{3}{x+1}$

- (a) Explain why $f(x)$ has an inverse function.
- (b) Find the equation of the inverse function.
- (c) Write down the domain and range of the inverse function.

**Example 41**

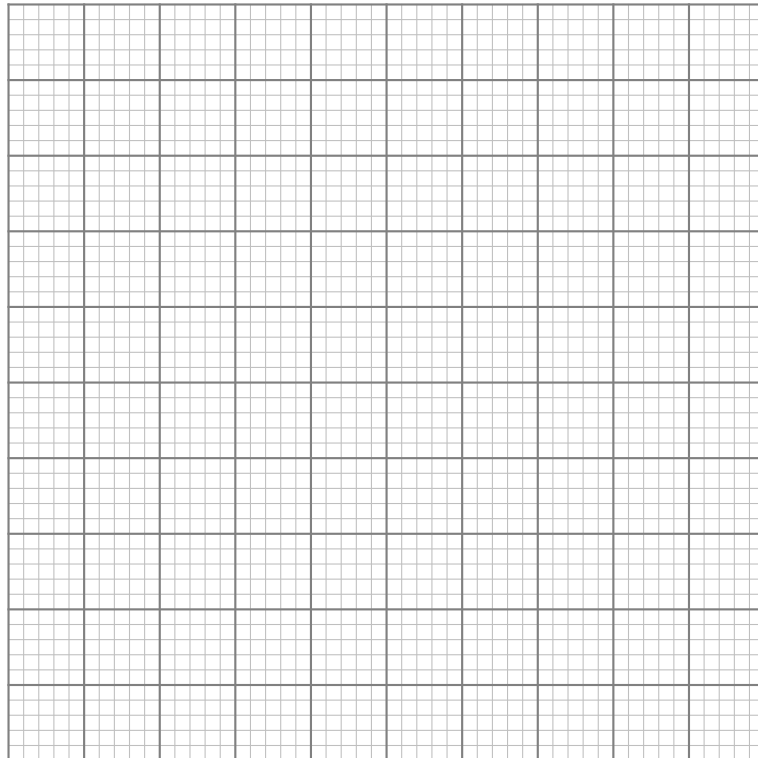
Find the inverse function of $y = x^3 - 2$, and sketch both curves on the same set of axes.



**Example 42**

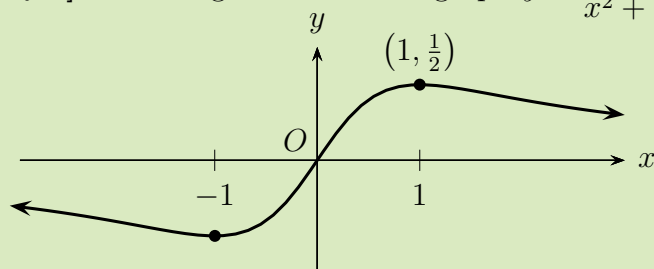
For the function $y = x^2 + 2x + 3$:

- (a) State its domain and range.
- (b) Sketch the curve $y = f(x)$.
- (c) Explain why the inverse function does not exist.
- (d) Restrict the domain such that the curve is monotonically increasing, sketch the inverse $f^{-1}(x)$ on the same axes as $f(x)$, and find its equation.



**Example 43**

[2018 Ext 1 HSC Q13] The diagram shows the graph $y = \frac{x}{x^2 + 1}$, for all real x .



Consider the function $f(x) = \frac{x}{x^2 + 1}$, for $x \geq 1$.

The function $f(x)$ has an inverse. (Do NOT prove this.)

- | | | |
|------|---|----------|
| i. | State the domain and range of $f^{-1}(x)$. | 2 |
| ii. | Sketch the graph $y = f^{-1}(x)$. | 1 |
| iii. | Find an expression for $f^{-1}(x)$. | 3 |

**Example 44**

[1996 3U HSC Q7] Consider the function $f(x) = \frac{1}{4}[(x-1)^2 + 7]$.

- (i) Sketch the parabola $y = f(x)$, showing clearly any intercepts with the axes, and the coordinates of its vertex. Use the same scale on both axes. **2**
- (ii) What is the largest domain containing the value $x = 3$, for which the function has an inverse function $f^{-1}(x)$? **1**
- (iii) Sketch the graph of $y = f^{-1}(x)$ on the same set of axes as your graph in part (i). Label the two graphs clearly. **2**
- (iv) What is the domain of the inverse function? **1**
- (v)  Let a be a real number not in the domain found in part (ii). Find $f^{-1}(f(a))$. **2**
- (vi) Find the coordinates of any points of intersection of the two curves $y = f(x)$ and $y = f^{-1}(x)$. **2**

 Further exercises**Ex 5F**

- All questions

Ex 5G

- Q1-9

Additional questions

- Find the inverse function of each of the following functions:
 - $y = x + 1$
 - $y = 3x - 2$
 - $y = \frac{x+2}{3}$
 - $y = x^3$
 - $y = (x+1)^3$
 - $y = \frac{1}{x-1}$
 - $y = \frac{x}{x-1}$
- The function $y = x$ is *invariant under inversion*, i.e. the equation of the function and its inverse are the same.
 - Give two more examples of functions that are invariant under inversion.
 - What do you notice about the the graphs of these functions?
- Sketch the graph of $y = \sqrt{3-x}$.
 - On the same axes, sketch the graph of its inverse function.
 - Find the equation of the inverse function.
 - Find the coordinates of the point of intersection of the function and its inverse.
- Show that the following pairs of functions are inverses by showing that over the restricted domain,

$$f \circ g(x) = g \circ f(x) = x$$

- $$\begin{cases} f(x) = 2x - 1 \\ g(x) = \frac{1}{2}(x + 1) \end{cases}$$
- $$\begin{cases} f(x) = 2x - x^2 & x \geq 1 \\ g(x) = 1 + \sqrt{1-x} & x \leq 1 \end{cases}$$
- $$\begin{cases} f(x) = \sqrt{16-x^2} & x \in [-4, 0] \\ g(x) = -\sqrt{16-x^2} & x \in [0, 4] \end{cases}$$
- $$\begin{cases} f(x) = \frac{1}{2x-1} & x > \frac{1}{2} \\ g(x) = \frac{x+1}{2x} & x > 0 \end{cases}$$

5. [2024 NBHS Year 11 Ext 1 Task 3 Q12]

3

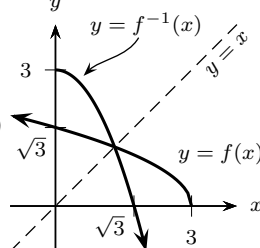
Consider the even function $f(x+a) = x^4$, where $a > 0$. $f^{-1}(x)$ is the inverse function of $f(x)$ and the range of $y = f^{-1}(x)$ is $y \leq a$.

If $c > a$, find the value of $f^{-1}[f(c)]$ in terms of a and c .

Answers

1. (a) $y = x - 1$ (b) $y = \frac{x+2}{3}$ (c) $y = 3x - 2$ (d) $y = \sqrt[3]{x}$ (e) $y = \sqrt[3]{x} - 1$ (f) $y = 1 + \frac{1}{x}$ (g) $y = \frac{x}{x-1}$ 2. (a) $y = -x$ (or more generally, $y = k - x$, k being any constant); $y = \frac{1}{x}$ (or $y = \frac{k}{x}$); $y = \sqrt{k^2 - x^2}$ over $D = \{x : 0 \leq x \leq k\}$; plus an infinite

number of others. (b) Symmetrical about $y = x$. 3. (a)



(b) See previous. (c) $y = 3 - x^2$, $x \geq 0$

- (d) $\left(\frac{\sqrt{13}-1}{2}, \frac{\sqrt{13}-1}{2}\right)$ 4. $f^{-1}(f(c)) = 2a - c$

Section 8

Parametric representation



Learning Goal(s)

Knowledge

Parametric form of functions and relations

Skills

Converting Cartesian form to parametric form

Understanding

The additional parameter used to represent the curve in terms of x and y values

By the end of this section am I able to:

3.21 Parametric form of a function or relation

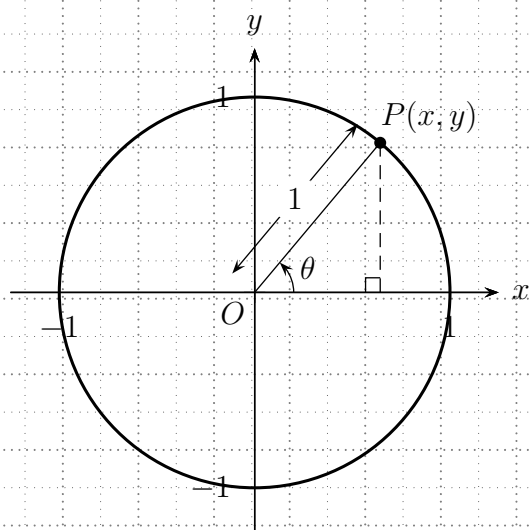
- Recognise that a curve may be represented by two parametric equations that give x and y as functions of a parameter and that the curve may also have a Cartesian equation in x and y .
- Express linear and quadratic functions and circles in parametric form.
- Convert linear and quadratic functions and circles from parametric form to Cartesian form.
- Graph linear and quadratic functions and circles expressed in parametric form.

8.1 Introduction



Example 45

Write down the equation of a circle with radius 1, centred at the origin in two different forms.



Definition 4



A *Cartesian* equation uses *rectangular coordinates* (x axis $\perp$ y axis). Named after René Descartes (1596–1650).

http://en.wikipedia.org/wiki/Rene_Descartes

- Other coordinate systems include
 - logarithmic $(x, \log_{10} x)$ – Richter scale, dB.
 - (x2) polar (r, θ) – navigation, bearings
 - cylindrical (3D polar) (ρ, θ, ϕ) – astronomy, cartography (GPS).

Definition 5

A *parametric* equation uses additional *parameter*(s) to represent the curve in terms of its x and y values.

 There may be more than one parametric representation for the same curve.

8.1.1 Common parameterisations

Circle $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$

- r :

Parabola $\begin{cases} x = 2at \\ y = at^2 \end{cases}$

- a :

8.1.2 Other parameterisations

- t for time in 2 dimensional motion, such as projectile motion.

$$\begin{cases} x = Vt \cos \alpha \\ y = -\frac{1}{2}gt^2 + Vt \sin \alpha \end{cases}$$

-  Previously in the legacy ‘4 Unit’ course:

Ellipse $\begin{cases} x = a \cos \theta \\ y = b \sin \theta \end{cases} \quad (a \neq b, a, b \neq 0)$

 a :

 b :

Hyperbola $\begin{cases} x = a \sec \theta \\ y = b \tan \theta \end{cases}$

 a :

Rectangular hyperbola $\begin{cases} x = ct \\ y = \frac{c}{t} \end{cases}$

8.2 Conversion from Parametric to Cartesian form

- Remove parameter by solving simultaneously



Example 46

Find the Cartesian equation of $\begin{cases} x = 3t + 1 \\ y = 2t - 3 \end{cases}$.

Answer: $2x - 3y - 11 = 0$

**Example 47**

Find the Cartesian equation of $\begin{cases} x = 2q \\ y = q^2 - 3 \end{cases}$.

Answer: $x^2 = 4y + 12$

Additional questions Source: [Fitzpatrick \(1984, Ex 24\(b\)\)](#)

Find the Cartesian equation of the curves whose parametric equations are:

1. $\begin{cases} x = 2t \\ y = t + 2 \end{cases} \quad (t \in \mathbb{R})$

6. $\begin{cases} x = v^3 \\ y = 1 - v^2 \end{cases} \quad (v \in [-1, 1])$

2. $\begin{cases} x = t \\ y = t^2 \end{cases} \quad (t \in \mathbb{R})$

7. $\begin{cases} x = t + 2 \\ y = t^2 - 1 \end{cases} \quad (t \in \mathbb{R})$

3. $\begin{cases} x = t \\ y = \frac{1}{t} \end{cases} \quad (t \in \mathbb{R})$

8. $\begin{cases} x = 2t^2 \\ y = 4t \end{cases} \quad (t \in \mathbb{R})$

4. $\begin{cases} x = t + 3 \\ y = t^2 - 5 \end{cases} \quad (t \geq 0)$

9. $\begin{cases} x = \frac{2t}{1+t^2} \\ y = \frac{1-t^2}{1+t^2} \end{cases} \quad (t \in \mathbb{R})$

5. $\begin{cases} x = 2u - 2 \\ y = 3u + 1 \end{cases} \quad (u \in [1, 3])$

Answers

1. $2y = x + 4, x \in \mathbb{R}$ 2. $y = x^2, x \in \mathbb{R}$ 3. $y = \frac{1}{x}, x \neq 0$ 4. $x^2 - 6x + 4, x \in [3, \infty)$ 5. $2y = 3x + 8, x \in [0, 4]$ 6. $y = 1 - x^{\frac{2}{3}}, x \in [-1, 1]$ 7. $y = x^2 - 4x + 3, x \in \mathbb{R}$ 8. $y^2 = 8x, x \in [0, \infty)$ 9. $x^2 + y^2 = 1, y \neq -1$

**Further exercises****Ex 5H**

- Q6-12

8.3 Other harder examples



Important note

Check the domain and/or range, or any other asymptotes that may or may not appear.



Example 48

[Ex 9D Q5] Find the Cartesian equation:
$$\begin{cases} x = t + \frac{1}{t} \\ y = t^2 + \frac{1}{t^2} \end{cases}$$

Answer: $y = x^2 - 2$, $x \leq -2$ or $x \geq 2$

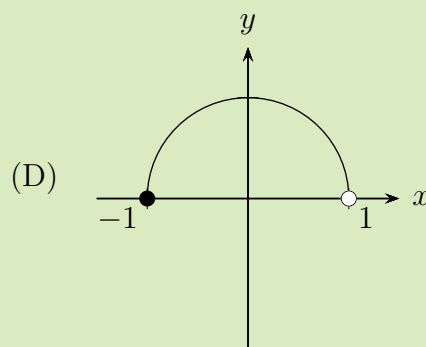
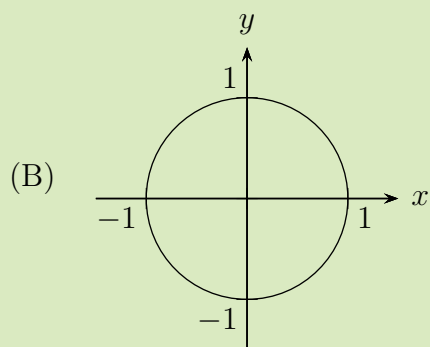
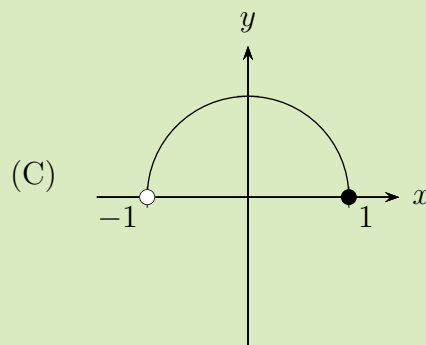
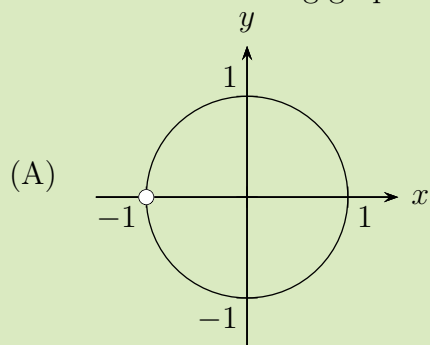
**Example 49**

[2025 CSSA Ext 1 Trial Q8] A curve is defined parametrically as

$$\begin{cases} x = \frac{1-t^2}{1+t^2} \\ y = \frac{2t}{1+t^2} \end{cases} \quad \text{for } t \in [0, \infty)$$

Which of the following graphs best represents the curve?

Answer: (C)



References

- Fitzpatrick, J. B. (1984). *New Senior Mathematics – Three Unit Course for Years 11 & 12*. Harcourt Education.
- Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019). *CambridgeMATHS Stage 6 Mathematics Extension 1 Year 11* (1st ed.). Cambridge Education.