NORMANHURST BOYS HIGH SCHOOL

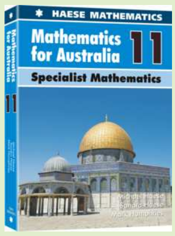
MATHEMATICS EXTENSION 1

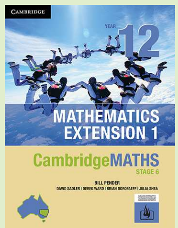
YEAR 11/12 COURSE

Topic summary and exercises:

Ⓧ1 **Introduction to Vectors**

With references to





Name: ....

Initial version by H. Lam, September 2019 with assistance from I. Ham. Revised by H. Lam for the 2026 new syllabuses.
Last updated March 17, 2026
Various corrections by students & members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at http://www.flaticon.com, used under CC BY 2.0.

Parts of this handout are derived from Haese, Haese, and Humphries (2015), *MATH1131 Mathematics 1A and MATH1141 Higher Mathematics 1A Algebra Notes* (2018)and Pender, Sadler, Ward, Dorofaeff, and Shea (2019). Most GeoGebra applets listed here are courtesy of the NSW Department of Education's Mathematics Curriculum Support team/mEsh project's efforts.

Symbols used

Beware! Heed warning.

Mathematics Extension 1 content, for MAX coded class

Literacy: note new word/phrase.

Facts/formulae to memorise.

Provided on NESA Reference Sheet

Further reading/exercises to enrich your understanding and application of this topic.

Facts/formulae to understand, as opposed to blatant memorisation.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Content outcomes addressed

ME1-12-02 Operates with 2D and 3D vectors and uses 2D vectors to solve problems involving motion in two dimensions

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Further exercises from
  - *Haese Mathematics for Australia - Specialist Mathematics 11* (Haese et al., 2015)
  - *CambridgeMATHS Year 12 Extension 1* (Pender et al., 2019)
  - *CambridgeMATHS Extension 2* (Sadler & Ward, 2019)

  will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Learning intentions & outcomes

✓ **Content**

☐ 5.1 Vector representation and notation

- Define a vector as a quantity having both magnitude and direction.
- Associate vectors with directed line segments and recognise that a vector may have many directed line segments associated with it.
- Identify and use notation for vectors in both two dimensions and three dimensions, including $\mathbf{a}$, $\mathbf{a}$, and $\overrightarrow{AB}$, where $\overrightarrow{AB}$ is the vector with magnitude and direction those of the directed line segment from A to B.
- Use notations $|\underset{\sim}{a}|$, $|\mathbf{a}|$, and $\left|\overrightarrow{AB}\right|$ to represent the magnitude of a vector.
- Describe a position vector as a vector with its tail at the origin.
- Represent vectors graphically with and without graphing applications.
- Recognise and use the fact that two vectors are equal if they have the same magnitude and direction to solve problems.

✓ Content

☐ 5.2 Introduction to 2D and 3D vectors

- Use Cartesian coordinates to represent points in 2-dimensional (2D) and 3-dimensional (3D) space with and without graphing applications.
- Use the midpoint and distance formulas in two dimensions and three dimensions.
- Identify that, in three dimensions, all points on the xy-plane have a z-coordinate of 0, and deduce similar properties for points on the xz and yz-planes and thus the equations of the three coordinate planes.
- Define the zero vector $\mathbf{0}$, written as $\underset{\sim}{0}$, as the vector with zero magnitude and no direction.
- Define unit vectors as vectors of magnitude 1.
- Recognise and use $\widehat{\underset{\sim}{a}}$ and $\widehat{\mathbf{a}}$ as the notation for the unit vector in the direction of $\mathbf{a}$.
- Define the standard perpendicular unit vectors $\mathbf{i}$ and $\mathbf{j}$ in two dimensions and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in three dimensions.
- Express 2D vectors in component form, $x\,\mathbf{i} + y\,\mathbf{j}$; as an ordered pair, (x, y); and in column vector notation $\begin{pmatrix} x \\ y \end{pmatrix}$.
- Recognise $\overrightarrow{AB} = \begin{pmatrix} u-x \\ v-y \end{pmatrix}$ as the vector associated with the directed line segment from the point $A(x, y)$ to $B(u, v)$ in two dimensions.
- Express 3D vectors in component form, $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; as an ordered triple, (x, y, z); and in column vector notation $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
- Recognise $\overrightarrow{AB} = \begin{pmatrix} u-x \\ v-y \\ w-z \end{pmatrix}$ as the vector associated with the directed line segment from $A(x, y, z)$ to $B(u, v, w)$ in three dimensions.

✓ Content

☐ 5.3 Operating with vectors

- Define a scalar as a real number that is used to multiply a vector.
- Represent geometrically a scalar multiple of a vector in two dimensions and three dimensions with and without graphing applications.
- Perform multiplication of a vector by a scalar algebraically in component form.
- Establish and identify $\mathbf{a} = k\mathbf{b}$, for a non-zero scalar k, as a condition for two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ to be parallel to each other and determine with justification if two vectors are parallel to one another.
- Identify $\begin{pmatrix} a \\ -b \end{pmatrix}$ and $\begin{pmatrix} -a \\ b \end{pmatrix}$ as vectors perpendicular to $\begin{pmatrix} a \\ b \end{pmatrix}$ and with equal magnitude.
- Perform addition and subtraction of vectors algebraically in component form, and verify, with and without graphing applications, that geometrically these are obtained using the triangle law or the parallelogram law.
- Establish and calculate the magnitude of a vector using $|x\,\mathbf{i} + y\,\mathbf{j}| = \sqrt{x^2 + y^2}$ for 2D vectors and $|x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$ for 3D vectors.
- Use the magnitude of a vector to find the unit vector $\widehat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$ in two dimensions and three dimensions.

✓ 🎓 **Content**

☐ 5.4 Further operations with vectors

- Define $\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2$ as the scalar (dot) product of vectors $\mathbf{a} = x_1\,\mathbf{i} + y_1\,\mathbf{j}$ and $\mathbf{b} = x_2\,\mathbf{i} + y_2\,\mathbf{j}$ and use the scalar product to solve problems.
- Define $\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$ as the scalar product of vectors $\mathbf{a} = x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k}$ and $\mathbf{b} = x_2\,\mathbf{i} + y_2\,\mathbf{j} + z_2\,\mathbf{k}$ and use the scalar product to solve problems.
- Use $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ as a geometric expression of the scalar product of non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ in two dimensions and three dimensions, where θ is the angle between the vectors and $0^\circ \leq \theta \leq 180^\circ$.
- Verify the equivalence of $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ with the algebraic definition of the scalar product, $\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2$ for two dimensions and $\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$ for three dimensions.
- Derive and use the property $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ to establish the scalar product definition of the magnitude of a vector ($|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$) in two dimensions and three dimensions.
- Calculate the angle between two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, in both two dimensions and three dimensions, using the scalar product by deriving and applying the relationship $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \widehat{\mathbf{a}} \cdot \widehat{\mathbf{b}}$.
- Establish $\mathbf{a} \cdot \mathbf{b} = 0$ as a condition for two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ to be perpendicular to each other and use it to determine if two vectors are perpendicular.
- Establish $\mathbf{a} \cdot \mathbf{b} = \pm\,|\mathbf{a}|\,|\mathbf{b}|$ as another way to determine if two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel.
- Define the projection of a vector $\mathbf{a}$ onto a vector $\mathbf{b}$, denoted by $\mathrm{proj}_{\mathbf{b}}\,\mathbf{a}$, to be the vector component of $\mathbf{a}$ in the direction of vector $\mathbf{b}$.
- Examine the proof of the formula $\mathrm{proj}_{\mathbf{b}}\,\mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \left(\mathbf{a} \cdot \widehat{\mathbf{b}}\right)\widehat{\mathbf{b}}$ and use the formula to solve problems.
- Determine that the component of a vector $\mathbf{a}$ perpendicular to another vector $\mathbf{b}$ is $\mathbf{a} - \mathrm{proj}_{\mathbf{b}}\,\mathbf{a}$.

Contents

Part I

Vectors in two dimensions

Section 1

Introduction to Vectors

Learning Goal(s)

Knowledge
What is a vector

Skills
Operate with vectors and vector equations

Understanding
Difference between scalars and vectors

By the end of this section am I able to:

5.1 Vector representation and notation

- Define a vector as a quantity having both magnitude and direction.
- Associate vectors with directed line segments and recognise that a vector may have many directed line segments associated with it.
- Identify and use notation for vectors in both two dimensions and three dimensions, including $\mathbf{a}$, $\underset{\sim}{a}$, and $\overrightarrow{AB}$, where $\overrightarrow{AB}$ is the vector with magnitude and direction those of the directed line segment from A to B.
- Use notations $|\underset{\sim}{a}|$, $|\mathbf{a}|$, and $\left|\overrightarrow{AB}\right|$ to represent the magnitude of a vector.
- Describe a position vector as a vector with its tail at the origin.
- Represent vectors graphically with and without graphing applications.
- Recognise and use the fact that two vectors are equal if they have the same magnitude and direction to solve problems.

5.2 Introduction to 2D and 3D vectors

- Use Cartesian coordinates to represent points in 2-dimensional (2D) and 3-dimensional (3D) space with and without graphing applications.
- Use the midpoint and distance formulas in two dimensions and three dimensions.
- Identify that, in three dimensions, all points on the xy-plane have a z-coordinate of 0, and deduce similar properties for points on the xz and yz-planes and thus the equations of the three coordinate planes.
- Define the zero vector $\mathbf{0}$, written as $\underset{\sim}{0}$, as the vector with zero magnitude and no direction.
- Define unit vectors as vectors of magnitude 1.
- Recognise and use $\hat{\underset{\sim}{a}}$ and $\hat{\mathbf{a}}$ as the notation for the unit vector in the direction of $\mathbf{a}$.
- Define the standard perpendicular unit vectors $\mathbf{i}$ and $\mathbf{j}$ in two dimensions and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in three dimensions.
- Express 2D vectors in component form, $x\,\mathbf{i} + y\,\mathbf{j}$; as an ordered pair, (x, y); and in column vector notation $\begin{pmatrix} x \\ y \end{pmatrix}$.
- Recognise $\overrightarrow{AB} = \begin{pmatrix} u-x \\ v-y \end{pmatrix}$ as the vector associated with the directed line segment from the point $A(x, y)$ to $B(u, v)$ in two dimensions.
- Express 3D vectors in component form, $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; as an ordered triple, (x, y, z); and in column vector notation $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
- Recognise $\overrightarrow{AB} = \begin{pmatrix} u-x \\ v-y \\ w-z \end{pmatrix}$ as the vector associated with the directed line segment from $A(x, y, z)$ to $B(u, v, w)$ in three dimensions.

1.1 Definition

Definition 1

Quantities which only have .... are known as *scalars*.

Example: .... of an aeroplane.

Definition 2

Quantities which have both .... and .... are known as *vectors*.

Example: .... of an aeroplane.

Fill in the spaces

Other examples of vector forces:

- ....
- ....
- ....
- ....

1.2 Representation

1.2.1 Geometric

Fill in the spaces

Vector quantities can be represented using a *directed line segment* or *arrow*.

- The .... of the arrow represents the ....
- The .... shows its direction.

Definition 3

Two vectors are *equal* if they have the same .... and ....

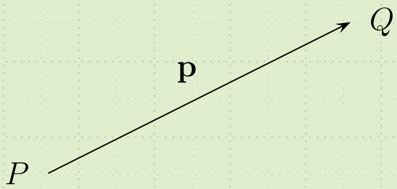
Corollary 1

The position of the starting point (tail) and ending point (arrowhead) does .... matter.

Example 1

Given $\overrightarrow{PQ}$ as shown, draw two other vectors, $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{e} = \overrightarrow{EF}$ such that

$$\overrightarrow{PQ} = \overrightarrow{AB} = \overrightarrow{EF}$$



1.2.2 Algebraic

Fill in the spaces

From Example 1, there are three ways of writing the vector drawn:

- $\overrightarrow{PQ}$: "arrow vector notation" or "double letter notation". The vector .... from the point P and .... at point Q.
- $\mathbf{p}$, $\underset{\sim}{p}$, $\vec{p}$: "single letter notation". The **boldface** type appears in typeset documents, the latter two are generally handwritten.

1.3 Vector arithmetic

1.3.1 Addition

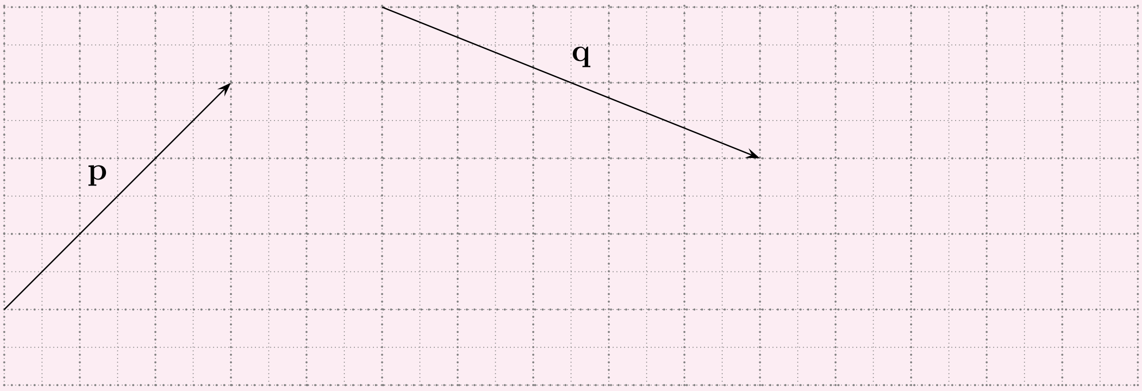
Important note

Use a ruler!

Steps

For vectors $\mathbf{p}$ and $\mathbf{q}$, to perform $\mathbf{p}+\mathbf{q}$:

1. Draw $\mathbf{p}$.
2. Draw $\mathbf{q}$, starting from the arrowhead of $\mathbf{p}$.
3. Draw .... vector from the .... of $\mathbf{p}$ to the .... of $\mathbf{q}$.



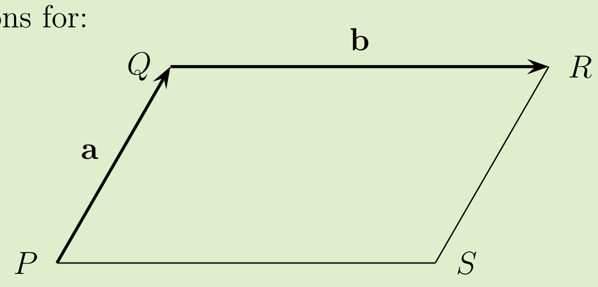
1.3.2 Subtraction/negative vectors

Definition 4

If $\mathbf{p}$ is a vector, then $-\mathbf{p}$ is equal in .... but .... in ....

Example 2

(Haese et al., 2015) $PQRS$ is a parallelogram where $\overrightarrow{PQ}=\mathbf{a}$ and $\overrightarrow{QR}=\mathbf{b}$. Find vector expressions for:



(a) $\overrightarrow{QP}$ (b) $\overrightarrow{RQ}$ (c) $\overrightarrow{SR}$ (d) $\overrightarrow{SP}$

1.3.3 Magnitude and zero vector

Definition 5

The *magnitude* of a vector $\mathbf{p}$ is given by $|\mathbf{p}|$.

Definition 6

The *zero vector* has magnitude zero, i.e. $|\mathbf{p}| = 0$. Direction is not ....

Usually written as $\mathbf{0}$ (handwritten: $\underset{\sim}{0}$).

1.3.4 Scalar multiplication

Definition 7

If $\mathbf{p}$ is a vector, and $\lambda \in \mathbb{R}$,

- If $\lambda > 1$, then $\lambda\mathbf{p}$ extends the .... to $\lambda\,|\mathbf{p}|$. (...................... the vector $\mathbf{p}$ by a factor of λ).
- If $0 < \lambda < 1$, then $\lambda\mathbf{p}$ .... the magnitude.
- If $\lambda = 0$, then $\lambda\mathbf{p} = \mathbf{0}$.
- If $\lambda < 0$, then $\lambda\mathbf{p}$ sees the magnitude scaled by $|\lambda|$ and direction ....

Laws/Results

- $\mathbf{p}$ and $\lambda\mathbf{p}$, $\lambda > 0$ are ....
- $\mathbf{p}$ and $\lambda\mathbf{p}$, $\lambda < 0$ are ....

GeoGebra

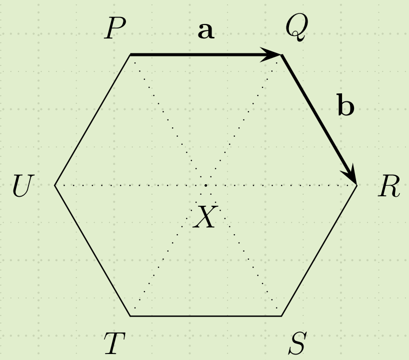
Scaling vector investigation

GeoGebra

Vector addition - what shape is *always* formed when adding vectors?

Example 3

(Haese et al., 2015) $PQRSTU$ is a regular hexagon. If $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{QR} = \mathbf{b}$, find in terms of $\mathbf{a}$ and $\mathbf{b}$:



(a) $\overrightarrow{PX}$

(b) $\overrightarrow{PS}$

(c) $\overrightarrow{QX}$

(d) $\overrightarrow{RS}$

1.4 Vector equations

Laws/Results

Vectors can be

- Added
- Have scalar multiplication applied

in the same manner as usual scalar numbers, with the same commutative/associative/distributive laws.

Example 4

(*MATH1131 Mathematics 1A and MATH1141 Higher Mathematics 1A Algebra Notes*, 2018) Fully simplify: $3(2\mathbf{a} - \mathbf{b}) + (\mathbf{a} - 2\mathbf{b})$.

Example 5

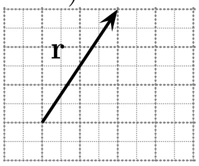
(*MATH1131 Mathematics 1A and MATH1141 Higher Mathematics 1A Algebra Notes*, 2018) Fully simplify: $2\overrightarrow{AC} - \overrightarrow{OC} + \overrightarrow{OA}$.

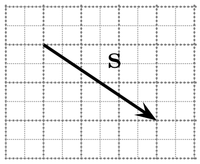
Important note

- Draw picture if it's slightly uncertain what is $\overrightarrow{AC}$.
- Convert the vector arrow notation $\overrightarrow{OC}$ to the single variable notation $\mathbf{c}$.

1.4.1 Additional questions

1. (Haese et al., 2015)

Given vectors 

and 

(diagrams drawn to scale), construct geometrically:

(a) $-\mathbf{r}$

(b) $2\mathbf{s}$

(c) $\frac{1}{2}\mathbf{r}$

(d) $-\frac{3}{2}\mathbf{s}$

(e) $2\mathbf{r} - \mathbf{s}$

(f) $2\mathbf{r} + 3\mathbf{s}$

(g) $\frac{1}{2}\mathbf{r} + 2\mathbf{s}$

(h) $\frac{1}{2}(\mathbf{r} + 3\mathbf{s})$

Further exercises

Ex 3A-3C (Haese et al., 2015)
(More introductory type questions)
- Every second subpart

Ex 8A (Pender et al., 2019)
- Q1-19

Section 2

Two dimensional vector geometry

Learning Goal(s)

Knowledge
Vector operations

Skills
Identifying parallel and perpendicular vectors, adding/subtracting vectors, finding magnitude

Understanding
Unit vectors being an application of finding the magnitude of vector

By the end of this section am I able to:

5.3 Operating with vectors

- Define a scalar as a real number that is used to multiply a vector.
- Represent geometrically a scalar multiple of a vector in two dimensions and three dimensions with and without graphing applications.
- Perform multiplication of a vector by a scalar algebraically in component form.
- Establish and identify $\mathbf{a} = k\mathbf{b}$, for a non-zero scalar k, as a condition for two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ to be parallel to each other and determine with justification if two vectors are parallel to one another.
- Identify $\begin{pmatrix} a \\ -b \end{pmatrix}$ and $\begin{pmatrix} -a \\ b \end{pmatrix}$ as vectors perpendicular to $\begin{pmatrix} a \\ b \end{pmatrix}$ and with equal magnitude.
- Perform addition and subtraction of vectors algebraically in component form, and verify, with and without graphing applications, that geometrically these are obtained using the triangle law or the parallelogram law.
- Establish and calculate the magnitude of a vector using $|x\,\mathbf{i} + y\,\mathbf{j}| = \sqrt{x^2 + y^2}$ for 2D vectors and $|x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$ for 3D vectors.
- Use the magnitude of a vector to find the unit vector $\widehat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$ in two dimensions and three dimensions.

2.1 Vector components

2.1.1 Unit vector

Definition 8

The *unit vector* has .... 1.

Laws/Results

For every vector $\mathbf{p}$ there are two *normalised* (corresponding unit) vectors:

- $\widehat{\mathbf{p}} = \dfrac{\mathbf{p}}{|\mathbf{p}|}$. .... to $\mathbf{p}$.
- $-\widehat{\mathbf{p}} = -\dfrac{\mathbf{p}}{|\mathbf{p}|}$. .... to $\mathbf{p}$.

GeoGebra

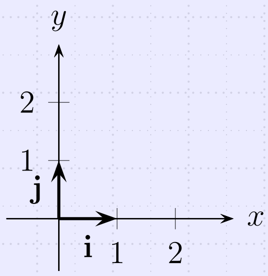
Unit vector

2.1.2 Cartesian basis vectors/components

Definition 9

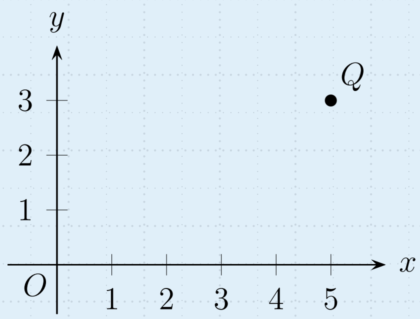
In the Cartesian plane, define the *basis vectors* to be unit vectors in the

- x direction to be $\mathbf{i}$
- y direction to be $\mathbf{j}$



Fill in the spaces

- The position $(5, 3)$ can now be represented via a *translation vector*.



- .... notation:

$$\overrightarrow{OQ} = \text{..............}$$

- .... notation:

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Definition 10

The point $Q(x, y)$ has *position vector*

$$\overrightarrow{OQ} = \begin{pmatrix} x \\ y \end{pmatrix} = x\,\mathbf{i} + y\,\mathbf{j}$$

where

- The *component form* is $x\,\mathbf{i} + y\,\mathbf{j}$
- Represented in .... notation: $\begin{pmatrix} x \\ y \end{pmatrix}$.
  - Pender et al. (2019) and some other publications uses square brackets instead of parentheses for the column vector delimiters, e.g.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- Year 7-10 *ordered pair* notation: (x, y).

Example 6

A ship S leaves port O and sails north-east at 20 km/h. Describe its position after 6 hours by giving its position vector as a sum of multiples of the basis vectors $\mathbf{i}$ and $\mathbf{j}$.

2.1.3 Position vs displacement vectors

Definition 11

A *position vector* of a point P is the vector $\overrightarrow{OP}$, where the vector commences at the .... and finishes at P.

A position vector gives the .... of a point in the plane.

- Each point P in the Cartesian plane corresponds to the position vector ....

Definition 12

A *displacement vector* is the vector $\overrightarrow{AB}$, where the vector commences at the point A and finishes at B.

A displacement vector gives the .... of a point from another.

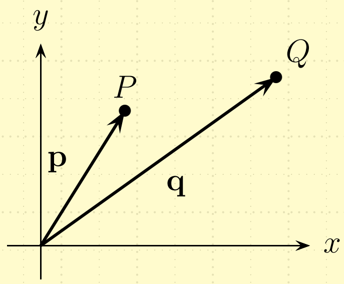
- Vector of point A relative to point B: ....

GeoGebra

Position Vector Investigation - see how a position vector is different to a displacement vector.

Laws/Results

For two points P and Q with position vectors $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ respectively,



$$\overrightarrow{OP} + \overrightarrow{PQ} = \ldots\ldots\ldots$$

$$\overrightarrow{PQ} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots$$

$$= \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix}$$

$$\overrightarrow{OQ} + \overrightarrow{QP} = \ldots\ldots\ldots$$

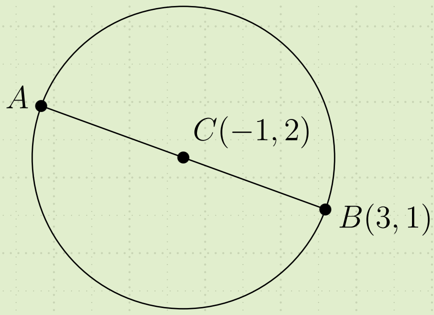
$$\overrightarrow{QP} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots$$

$$= \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix}$$

Example 7

(Haese et al., 2015) AB is a diameter of a circle with centre $C(-1, 2)$. If B is $(3, 1)$, find:



(a) $\overrightarrow{BC}$

(b) the coordinates of A.

2.2 Vector arithmetic involving column vectors

2.2.1 Addition

Laws/Results

If $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} u \\ v \end{pmatrix}$, then

$$\mathbf{p} \pm \mathbf{q} = \dots\dots\dots\dots\dots\dots\dots$$

$$= \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Important note

- Add .... components (x direction, $\mathbf{i}$) with .... components
- Add .... components (y direction, $\mathbf{j}$) with .... components

GeoGebra

Vector component addition

Example 8

(Pender et al., 2019)

1. (a) Given $\mathbf{u} = 2\,\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 2\,\mathbf{j}$, find:

i. $2\mathbf{u}$ ii. $3\mathbf{v}$ iii. $2\mathbf{u} + 3\mathbf{v}$

(b) Draw all five vectors as position vectors. What figure do the origin and heads of $2\mathbf{u}$, $3\mathbf{v}$ and $2\mathbf{u} + 3\mathbf{v}$ form?

2. (a) Given $\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find $\mathbf{v} - \mathbf{u}$.

(b) Draw $\mathbf{u} = \overrightarrow{OU}$ and $\mathbf{v} = \overrightarrow{OV}$ as position vectors, and mark $\mathbf{v} - \mathbf{u}$.

2.2.2 Scalar multiplication

Laws/Results

(Haese et al., 2015) If $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $k \in \mathbb{R}$ then

$$k\mathbf{p} = \ldots\ldots\ldots\ldots\ldots$$

Example 9

Given $\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find:

(a) $3\mathbf{q}$ (b) $\mathbf{p} + 2\mathbf{q}$ (c) $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

2.2.3 Midpoint

Definition 13

Midpoint formula to find the midpoint M between two points $A(x_1, y_1)$ and $B(x_2, y_2)$:

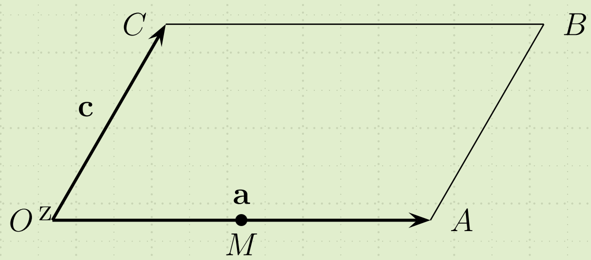
$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

For vectors $\overrightarrow{OA} = \mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\overrightarrow{OB} = \mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, the position vector for the midpoint M of AB is

$$\overrightarrow{OM} = \mathbf{m} = \frac{1}{2}\left(\ldots\ldots\ldots\ldots\right)$$

Example 10

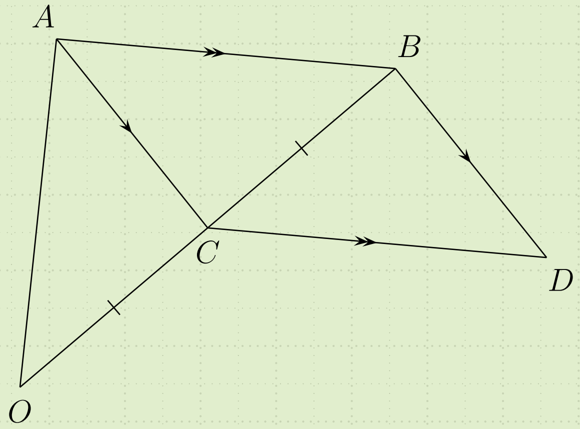
[2020 Independent Ext 1 Trial Q3] In the diagram, $OABC$ is a paralleogram. The vector $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. M is the midpoint of OA. Which of the following expressions is represented by the vector $\overrightarrow{MB}$?



(A) $\frac{1}{2}\mathbf{a} - \mathbf{c}$ (B) $\frac{1}{2}\mathbf{a} + \mathbf{c}$ (C) $\mathbf{a} - \frac{1}{2}\mathbf{c}$ (D) $\mathbf{a} + \frac{1}{2}\mathbf{c}$

Example 11

[2020 NEAP Ext 1 Trial Q7] The position vectors of points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C is the midpoint of OB and point D is such that $ABCD$ is a parallelogram.



Which of the following is the position vector of D?

(A) $\frac{3}{2}\mathbf{b}+\mathbf{a}$ (B) $\frac{3}{2}\mathbf{b}-\mathbf{a}$ (C) $\frac{1}{2}\mathbf{b}-\frac{1}{2}\mathbf{a}$ (D) $\frac{1}{2}\mathbf{b}-\mathbf{a}$

2.2.4 Magnitude

Definition 14

If $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$$|\mathbf{p}| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Example 12

If $\mathbf{p} = 3\,\mathbf{i} - 5\,\mathbf{j}$ and $\mathbf{q} = -\mathbf{i} - 2\,\mathbf{j}$, find $|\mathbf{p} - 2\mathbf{q}|$. **Answer:** $\sqrt{26}$

Example 13

(Haese et al., 2015, Ex 3D) Find k such that the vector is a unit vector.

(a) $\begin{pmatrix} 0 \\ k \end{pmatrix}$ (b) $\begin{pmatrix} k \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} k \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} k \\ k \end{pmatrix}$ (e) $\begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix}$

Answer: (a) ± 1 (b) ± 1 (c) 0 (d) $\pm\frac{1}{\sqrt{2}}$ (e) $\pm\frac{\sqrt{3}}{2}$

Example 14

(Haese et al., 2015, Ex 3D) Given $\mathbf{v} = \begin{pmatrix} 8 \\ p \end{pmatrix}$ and $|\mathbf{v}| = \sqrt{73}$, find the possible values of p.

Answer: ± 3

Example 15

(Haese et al., 2015) If $\mathbf{a} = 3\,\mathbf{i} - \mathbf{j}$, find:

(a) a unit vector in the direction of $\mathbf{a}$

(b) a vector of length 4 units in the direction of $\mathbf{a}$

(c) vectors of length 4 units which are parallel to $\mathbf{a}$.

Answer: (a) $\frac{3}{\sqrt{10}}\,\mathbf{i} - \frac{1}{\sqrt{10}}\,\mathbf{j}$ (b) $\frac{12}{\sqrt{10}}\,\mathbf{i} - \frac{4}{\sqrt{10}}\,\mathbf{j}$ (c) $\pm\left(\frac{12}{\sqrt{10}}\,\mathbf{i} - \frac{4}{\sqrt{10}}\,\mathbf{j}\right)$

Example 16

[2020 NSBHS Ext 1 Trial] $\overrightarrow{OM} = 2\,\mathbf{i} + 5\,\mathbf{j}$ and $\overrightarrow{OT} = -3\,\mathbf{i} + 7\,\mathbf{j}$. What is the value of $\left|\overrightarrow{MT}\right|$?

(A) 14 (B) $\sqrt{98}$ (C) $\sqrt{29}$ (D) $\sqrt{145}$

2.2.5 Direction

Laws/Results

If $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the direction of the vector is measured by the .... formed by the vector with the positive direction of the horizontal axis, i.e.

$$\tan\theta = \frac{y}{x}$$

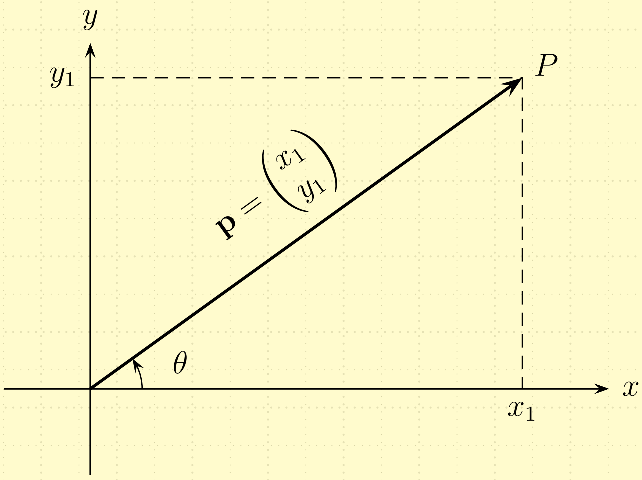
where $\theta \in [0°, 360°)$ (and later, radians where applicable)

Important note

X2 Later in Complex Numbers, $\theta \in (-\pi, \pi]$. The quadrants used may vary between contexts - read the context carefully.

Laws/Results

Components of a vector in terms of the magnitude and direction: if $\mathbf{p} = x_1\,\mathbf{i} + y_1\,\mathbf{j}$,



$x_1 =$ ....

$y_1 =$ ....

Example 17

(Pender et al., 2019)

(a) Find the vector **u** of length 10, with angle 150° to the positive direction of the horizontal axis.

(b) Find the length and angle (exact length, angle to the nearest degree) of $\mathbf{v} = -\mathbf{i} - 2\,\mathbf{j}$.

Answer: (a) $\mathbf{u} = -5\sqrt{3}\,\mathbf{i} + 5\,\mathbf{j}$(b) $|\mathbf{u}| = \sqrt{5}$, $\theta \approx 243°$.

Further exercises

Ex 3D-3G (Haese et al., 2015)
(More introductory type questions)

- Every second subpart

Ex 8B (Pender et al., 2019)

- Q11-18

Section 3

Vector Geometry with the Dot Product

Learning Goal(s)

Knowledge
What is the dot product

Skills
Linking various geometrical concepts with the dot product

Understanding
Dot product produces a scalar which then needs to be interpreted

By the end of this section am I able to:

5.4 Further operations with vectors

- Define $\mathbf{a}\cdot\mathbf{b} = x_1x_2 + y_1y_2$ as the scalar (dot) product of vectors $\mathbf{a} = x_1\,\mathbf{i} + y_1\,\mathbf{j}$ and $\mathbf{b} = x_2\,\mathbf{i} + y_2\,\mathbf{j}$ and use the scalar product to solve problems.
- Define $\mathbf{a}\cdot\mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$ as the scalar product of vectors $\mathbf{a} = x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k}$ and $\mathbf{b} = x_2\,\mathbf{i} + y_2\,\mathbf{j} + z_2\,\mathbf{k}$ and use the scalar product to solve problems.
- Use $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ as a geometric expression of the scalar product of non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ in two dimensions and three dimensions, where θ is the angle between the vectors and $0^\circ \leq \theta \leq 180^\circ$.
- Verify the equivalence of $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ with the algebraic definition of the scalar product, $\mathbf{a}\cdot\mathbf{b} = x_1x_2 + y_1y_2$ for two dimensions and $\mathbf{a}\cdot\mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$ for three dimensions.
- Derive and use the property $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$ to establish the scalar product definition of the magnitude of a vector ($|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$) in two dimensions and three dimensions.
- Calculate the angle between two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, in both two dimensions and three dimensions, using the scalar product by deriving and applying the relationship $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \widehat{\mathbf{a}}\cdot\widehat{\mathbf{b}}$.
- Establish $\mathbf{a}\cdot\mathbf{b} = 0$ as a condition for two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ to be perpendicular to each other and use it to determine if two vectors are perpendicular.
- Establish $\mathbf{a}\cdot\mathbf{b} = \pm|\mathbf{a}|\,|\mathbf{b}|$ as another way to determine if two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel.
- Define the projection of a vector $\mathbf{a}$ onto a vector $\mathbf{b}$, denoted by $\mathrm{proj}_{\mathbf{b}}\,\mathbf{a}$, to be the vector component of $\mathbf{a}$ in the direction of vector $\mathbf{b}$.
- Examine the proof of the formula $\mathrm{proj}_{\mathbf{b}}\,\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \left(\mathbf{a}\cdot\widehat{\mathbf{b}}\right)\widehat{\mathbf{b}}$ and use the formula to solve problems.
- Determine that the component of a vector $\mathbf{a}$ perpendicular to another vector $\mathbf{b}$ is $\mathbf{a} - \mathrm{proj}_{\mathbf{b}}\,\mathbf{a}$.

3.1 Definitions

Definition 15

The *dot product* between two vectors $\mathbf{p} = x_1\,\mathbf{i} + y_1\,\mathbf{j}$ and $\mathbf{q} = x_2\,\mathbf{i} + y_2\,\mathbf{j}$ (also known as *scalar product*):

$$\mathbf{p}\cdot\mathbf{q} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

GeoGebra

Dot Product Insight

Example 18

Use $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ etc to prove the following:

(a) $\mathbf{p} \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{p}$ (Commutativity)

(b) $\mathbf{p} \cdot \mathbf{p} = |\mathbf{p}|^2$

(c) $\mathbf{p} \cdot (\mathbf{q} + \mathbf{r}) = \mathbf{p} \cdot \mathbf{q} + \mathbf{p} \cdot \mathbf{r}$ (Associativity)

(d) $(\mathbf{p} + \mathbf{q}) \cdot (\mathbf{r} + \mathbf{s}) = \mathbf{p} \cdot \mathbf{r} + \mathbf{p} \cdot \mathbf{s} + \mathbf{q} \cdot \mathbf{r} + \mathbf{q} \cdot \mathbf{s}$

Example 19

(Haese et al., 2015, Ex 3I) Explain why $\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}$ is meaningless.

3.2 Angle between two vectors

The dot product enables the .... θ between two non-zero vectors to be found.

Definition 16

The angle θ between two non-zero vectors $\mathbf{p}$ and $\mathbf{q}$ is related by:

$$\therefore \mathbf{p} \cdot \mathbf{q} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Steps

Proof

1. Draw situation representing the angle θ between two vectors $\mathbf{p}$ and $\mathbf{q}$, and vector $\mathbf{q} - \mathbf{p}$.

2. Apply the cosine rule to the triangle:

$$|\mathbf{q} - \mathbf{p}|^2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

3. Let $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ and replace magnitudes with components of $\mathbf{p}$ and $\mathbf{q}$:

$$\therefore \mathbf{p} \cdot \mathbf{q} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

4. Consequently,

$$\cos\theta = \ldots\ldots\ldots\ldots$$

Laws/Results

Geometric results of the dot product

- If $\mathbf{p}$ is *perpendicular* to $\mathbf{q}$, then $\theta = 90^\circ$:

$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= |\mathbf{p}|\,|\mathbf{q}|\cos\theta \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= \ldots\end{aligned}$$

- If $\mathbf{p}$ is *parallel* to $\mathbf{q}$, then $\theta = 0^\circ$ or 180°:

$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= |\mathbf{p}|\,|\mathbf{q}|\cos\theta \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ or } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Corollary 2

If $\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}$, then the components of a vector $\mathbf{q}$ that is perpendicular to $\mathbf{p}$ are

$$\mathbf{q} = \ldots\ldots\ldots\ldots\ldots \text{ or } \ldots\ldots\ldots\ldots\ldots$$

(Verify this with the gradient of a line in the 2D Cartesian plane)

GeoGebra

Parallel Vectors investigation

Example 20

[2022 Ext HSC Q11] (2 marks) The vectors $\mathbf{u} = \begin{pmatrix} a \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} a-7 \\ 4a-1 \end{pmatrix}$ are perpendicular.

What are the possible values of a? **Answer:** 2 or $-\frac{7}{4}$

Example 21

(Haese et al., 2015) Consider the points $A(2,1)$ and $B(6,-1)$ and $C(5,-3)$. Use the dot product to determine whether $\triangle ABC$ is right angled or not. If so, locate the right angle.

Answer: At B

Example 22

(Haese et al., 2015) Find the form of *all* vectors which are perpendicular to $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Answer: $k\begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Example 23

(Haese et al., 2015) Find a vector of length 3 units which is perpendicular to $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

Answer: $\frac{3}{\sqrt{17}}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

Example 24

(Haese et al., 2015) Find the angle between $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ **Answer:** $\approx 116°$

Example 25

(Haese et al., 2015) Given $A(2, -1)$, $B(3, 4)$ and $C(-1, 3)$, find the size of $\angle ABC$.

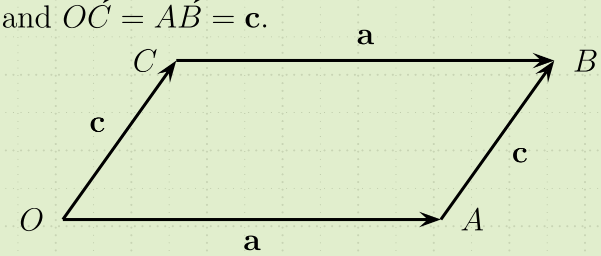
Answer: $\approx 64.7°$

Important note

Draw a simple diagram first!

Example 26

[2020 NEAP Ext 1 Trial Q10] The diagram shows $OABC$, a rhombus in which $\overrightarrow{OA} = \overrightarrow{CB} = \mathbf{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \mathbf{c}$.



To prove that the diagonals of $OABC$ are perpendicular, it is required to show that

(A) $(\mathbf{a}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{c}) = 0$

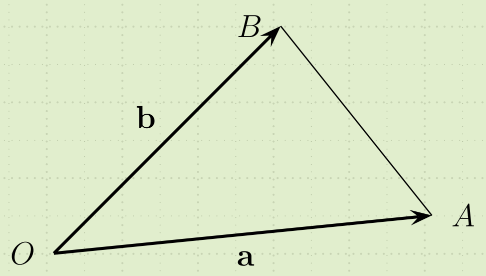
(B) $(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{c}) = 0$

(C) $(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}+\mathbf{c}) = 0$

(D) $\mathbf{a}\cdot\mathbf{c} = 0$

Example 27

[2020 Independent Ext 1 Trial Q11] In the diagram, OAB is an acute angled triangle in which $OA = OB$. The vector $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



(i) Show that $\cos \angle OAB = \dfrac{\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\ |\mathbf{a} - \mathbf{b}|}$ and $\cos \angle OBA = \dfrac{\mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\ |\mathbf{a} - \mathbf{b}|}$ **4**

(ii) Hence show that $\angle OAB = \angle OBA$ by using vector methods. **2**

Example 28

[2020 JRAHS Ext 1 Trial Q12] In the isosceles triangle ABC, $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AC}\right|$. D is the midpoint of side AB and E is the midpoint of side AC. $\overrightarrow{CD}$ is perpendicular to $\overrightarrow{BE}$.

Answer: $\cos\theta = \frac{4}{5} \Rightarrow \theta \approx 36°52'$

i. Draw the diagram and label $\angle BAC = \theta$ and $\left|\overrightarrow{AD}\right| = r$. **1**

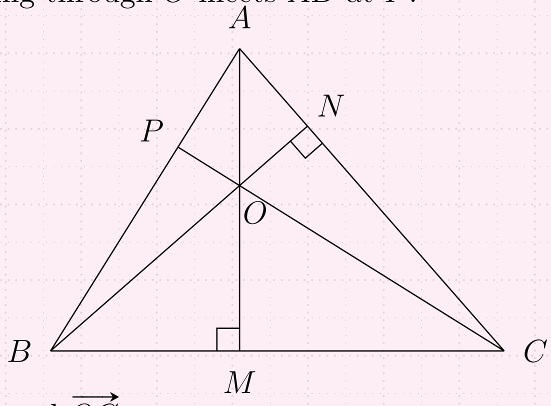
ii. Hence noting that $\overrightarrow{CD}$ may be written as $\overrightarrow{AD} - \overrightarrow{AC}$, or otherwise, use vector methods to find the value of $\angle BAC$. **3**

Timed exam practice 1 (Allow approximately 5 minutes)

[2024 Sydney Girls HS Ext 1 Trial Q14] (3 marks) In $\triangle ABC$, AM is perpendicular to BC and BN is perpendicular to AC.

Let O be the point of intersection of AM and BN.

The line from C passing through O meets AB at P.



Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

By using vector methods, show that CP is perpendicular to AB

Marking criteria

✓ [1] For recognising one of $\mathbf{b} \cdot (\mathbf{a} - \mathbf{c}) = 0$ or $\mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0$

✓ [2] For recognising both of $\mathbf{b} \cdot (\mathbf{a} - \mathbf{c}) = 0$ and $\mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0$

✓ [3] Correct solution

Further exercises

Ex 3I-3J (Haese et al., 2015)
(More introductory type questions)
- Every second subpart

Ex 8C (Pender et al., 2019)
- Q7-18

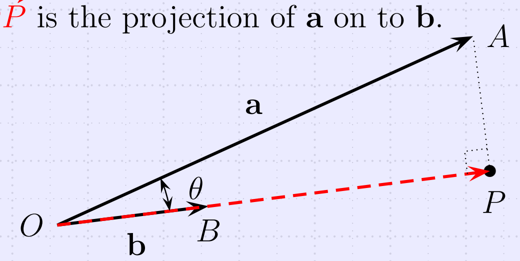
3.3 Projection of a vector on to another vector

Definition 17

The *projection* of a vector $\mathbf{a}$ on another vector $\mathbf{b}$ is a vector parallel to $\mathbf{b}$ with the following notation:

$$\text{proj}_{\mathbf{b}}\,\mathbf{a}$$

In the diagram given, $\overrightarrow{OP}$ is the projection of $\mathbf{a}$ on to $\mathbf{b}$.



Assumption: $\mathbf{b}$ is a ....

Important note

Plain English: the projection of vector $\mathbf{a}$ on to vector $\mathbf{b}$ is the .... from .... on to ....

Laws/Results

The projection of $\mathbf{a}$ on to $\mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$:

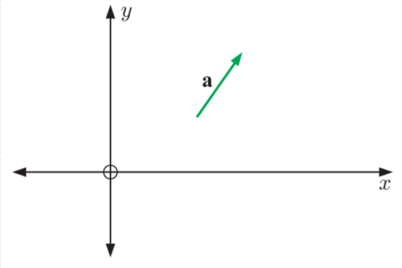
$$\text{proj}_{\mathbf{b}}\,\mathbf{a} = \ldots\ldots\ldots\ldots\ldots\ldots$$

GeoGebra

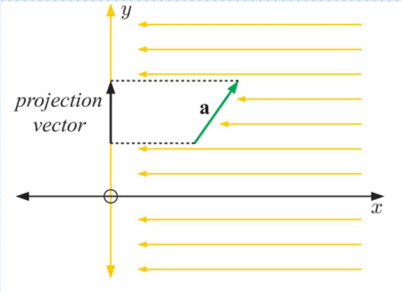
Projection of vector $\mathbf{a}$ on to $\mathbf{b}$

Fill in the spaces

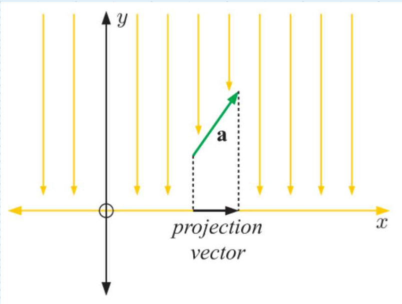
The following diagrams are taken from Haese et al. (2015), explanations paraphrased:



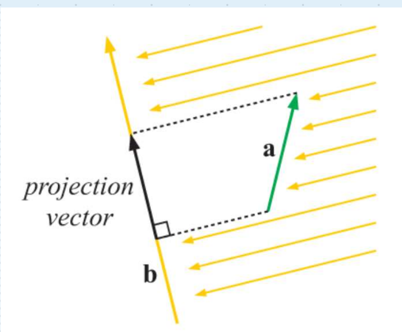
- Consider a vector **a** in the .... plane.



- If a light is shone on to the vector from the .... , it will cast a .... on the y axis.
- This .... is the .... of **a** on the y axis
- Also known as the y component of the vector **a**.
- If $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, then $\text{proj}_{\mathbf{j}}\, \mathbf{a} = \ldots$



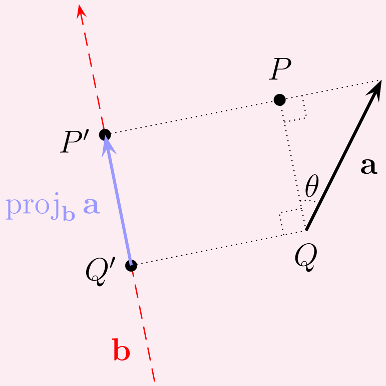
- Similarly, if a light is shone from the .... , the .... is the .... of **a** on the x axis.
- If $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, then $\text{proj}_{\mathbf{i}}\, \mathbf{a} = \ldots$



- If the light is shone through vector **a** on to another vector **b** so that the light is .... to **b**, the .... cast on **b** is the .... of **a** on to **b**.

Steps

Derivation Consider the diagram:



- Suppose θ is the acute angle between $\mathbf{a}$ and $\mathbf{b}$.
- Give the relationship between PQ, $|a|$ and θ:

$$\frac{PQ}{|\mathbf{a}|} = \ldots\ldots\ldots$$

$$\therefore PQ = \ldots\ldots\ldots\ldots\ldots\ldots$$

- As $\text{proj}_{\mathbf{b}}\,\mathbf{a}$ is a vector in the same .... to $\mathbf{b}$, it is also the unit vector in $\mathbf{b}$'s direction, multiplied by a scalar multiple:[a]

$$\begin{aligned}\text{proj}_{\mathbf{b}}\,\mathbf{a} &= \ldots\ldots\ \widehat{\mathbf{b}} \\ &= \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots}_{\text{scalar multiple, `magnitude'}}\ \widehat{\mathbf{b}} \qquad (\ddagger) \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ \frac{\mathbf{b}}{|\mathbf{b}|}\end{aligned}$$

- Recall also, that $\mathbf{a}\cdot\mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$, rearranging:

$$\cos\theta = \ldots\ldots\ldots\ldots$$

Substitute in the result from (‡):

$$\text{proj}_{\mathbf{b}}\,\mathbf{a} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

[a]See Section 2.1.1 on page 19

Laws/Results

The projection of vector $\mathbf{a}$ on to vector $\mathbf{b}$:

$$\text{proj}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = \underbrace{\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}}_{\text{scalar projection}}\ \widehat{\mathbf{b}}$$

Important note

- "Find" questions may still require proper reasoning to be provided.
- Read through your work carefully.

Example 29

Find the projection of $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ on to $\mathbf{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$, and state the length of the vector.

Answer: $\text{proj}_{\mathbf{b}}\, \mathbf{a} = \frac{9}{34}\begin{pmatrix} -3 \\ 5 \end{pmatrix}$, length $\frac{9}{\sqrt{34}}$

Example 30

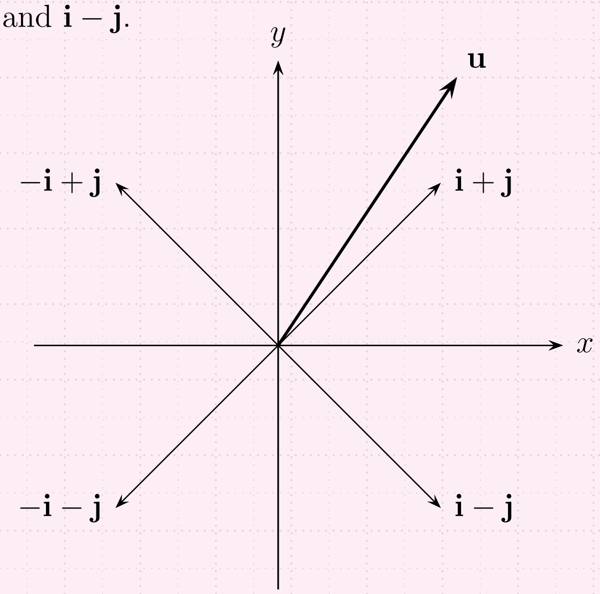
[**2020 JRAHS Ext 1 Trial Q11**] It is given that $A(-2,3), B(4,-5), C(-7,-6)$ and $D(-5,-2)$.

i. Find the vector projection of $\overrightarrow{AB}$ on to $\overrightarrow{CD}$. **2**

ii. What is the vector component of $\overrightarrow{AB}$ perpendicular to $\overrightarrow{CD}$? **1**

Timed exam practice 2 (Allow approximately 1 minute)

[**2022 Ext 1 HSC Q6**] The following diagram shows the vector $\mathbf{u}$ and the vectors $\mathbf{i}+\mathbf{j}$, $-\mathbf{i}+\mathbf{j}$, $-\mathbf{i}-\mathbf{j}$ and $\mathbf{i}-\mathbf{j}$.



(Drawn to scale)

Which statement regarding this diagram could be true? **Answer:** (B)

(A) The projection of $\mathbf{u}$ onto $\mathbf{i}+\mathbf{j}$ is the vector $1.1\,\mathbf{i}+1.8\,\mathbf{j}$.

(B) The projection of $\mathbf{u}$ onto $-\mathbf{i}+\mathbf{j}$ is the vector $-0.4\,\mathbf{i}+0.4\,\mathbf{j}$.

(C) The projection of $\mathbf{u}$ onto $-\mathbf{i}-\mathbf{j}$ is the vector $3.2\,\mathbf{i}+3.2\,\mathbf{j}$.

(D) The projection of $\mathbf{u}$ onto $\mathbf{i}-\mathbf{j}$ is the vector $0.5\,\mathbf{i}-0.5\,\mathbf{j}$.

Example 31

[**2025 Baulkham Hills HS Ext 1 Trial Q1**] Given that $\mathbf{u}=\begin{pmatrix}-2\\1\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}3\\4\end{pmatrix}$, which of the following best describes the direction of the projection of $\mathbf{u}$ on to $\mathbf{v}$?

Answer: (D)

(A) the projection is in the same direction as $\mathbf{u}$

(B) the projection is in the same direction as $\mathbf{v}$

(C) the projection is in the opposite direction to $\mathbf{u}$

(D) the projection is in the opposite direction to $\mathbf{v}$

Important note

....!

Example 32

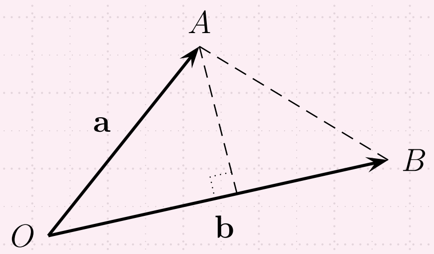
[2023 Ext 1 HSC Q6] Given the two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, let $\mathbf{c}$ be the projection of $\mathbf{a}$ onto $\mathbf{b}$.

What is the projection of $10\mathbf{a}$ onto $2\mathbf{b}$? **Answer:** (C)

(A) $2\mathbf{c}$ (B) $5\mathbf{c}$ (C) $10\mathbf{c}$ (D) $20\mathbf{c}$

Timed exam practice 3 (Allow approximately 8 minutes)

[2024 NBHS Year 11 Ext 1 Assessment Task 3 Q10] The diagram below shows $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. It is given that $\mathbf{a} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.



i. Find a vector that is perpendicular to $\mathbf{b}$. **1**

ii. Hence, or otherwise, find the shortest distance from the point A to the line OB by using vector methods. Leave your answer in exact form. **2**

iii. Find the area of the triangle OAB. **2**

Marking criteria

i. ✓ [1] Provides any vector that is a scalar multiple of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

ii. ✓ [1] For finding the projection of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ onto $\mathbf{a}$.

✓ [2] Provide the correct solution, $\frac{9}{\sqrt{5}}$

iii. ✓ [1] For a correct algebraic expression to $A_{\triangle OAB}$.

✓ [2] For the correct answer, $\frac{27}{2}$ units^2.

Timed exam practice 4 (Allow approximately 6 minutes)

[2024 Ext 1 HSC Q13] (4 marks) The vector $\mathbf{a}$ is $\begin{pmatrix}1\\3\end{pmatrix}$ and the vector $\mathbf{b}$ is $\begin{pmatrix}2\\-1\end{pmatrix}$.

The projection of a vector $\mathbf{x}$ onto the vector $\mathbf{a}$ is $k\mathbf{a}$, where k is a real number.

The projection of a vector $\mathbf{x}$ onto the vector $\mathbf{b}$ is $p\mathbf{b}$, where p is a real number.

Find the vector $\mathbf{x}$ in terms of k and p. **Answer:** $\mathbf{x} = \frac{5}{7}\begin{pmatrix}2k+3p\\4k-p\end{pmatrix}$

Marking criteria

✓ [1] Writes $\dfrac{\mathbf{a}\cdot\mathbf{x}}{\mathbf{a}\cdot\mathbf{a}}$ or $\dfrac{\mathbf{b}\cdot\mathbf{x}}{\mathbf{b}\cdot\mathbf{b}}$, or equivalent merit.

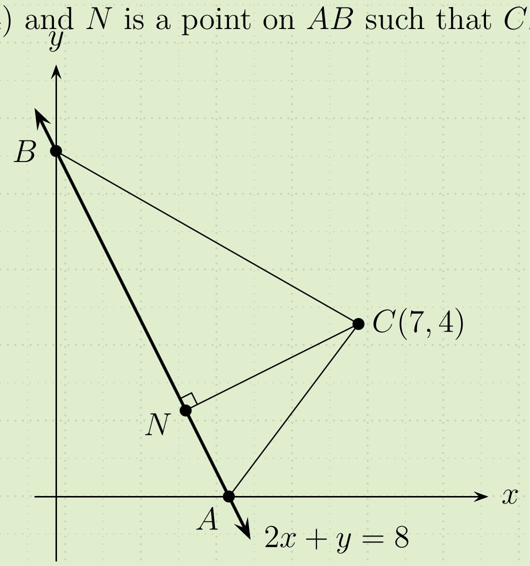
✓ [2] Obtains k or p in terms of the components of $\mathbf{x}$, or equivalent merit.

✓ [3] Obtains both k and p in terms of the components of $\mathbf{x}$, or equivelent merit.

✓ [4] Provides correct solution.

Example 33

[**2024 NBHS Year 12 Ext 1 Assessment Task 1 Q4**] The diagram shows $\triangle ABC$, and the line $2x + y = 8$ with x and y intercepts at A and B respectively. The point C has coordinates $(7, 4)$ and N is a point on AB such that $CN \perp AB$.



i. Show that $\overrightarrow{BA} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$. **2**

ii. By using vector methods, show that **2**

$$\cos \angle ABC = \frac{3}{\sqrt{13}}$$

iii. Find a vector $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ such that $\mathbf{u}$ is perpendicular to $\overrightarrow{BA}$. **1**

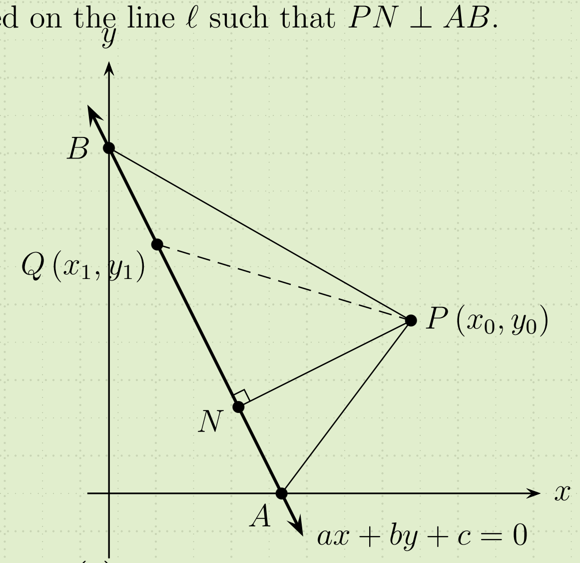
iv. Hence show that **2**

$$\overrightarrow{NC} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

v. Use vector methods to find the coordinates of N. **1**

Example 42

[2024 NBHS Year 12 Ext 1 Assessment Task 1 Q4] *(continued)* A line ℓ with equation $ax + by + c = 0$ is drawn in the Cartesian plane, and $P(x_0, y_0)$ is located off the line ℓ, and point Q has coordinates $Q(x_1, y_1)$ which lies on the line ℓ. N is a point that is located on the line ℓ such that $PN \perp AB$.



You are given that $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is a vector that is perpendicular to $\overrightarrow{BA}$ (Do NOT prove this).

vi. By finding $\overrightarrow{QP}$ using the coordinates given, show that **3**

$$\overrightarrow{NP} = \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}}\widehat{\mathbf{v}}$$

vii. Briefly **explain** why the shortest distance from a point (x_0, y_0) to a line $ax + by + c = 0$ is given by **1**

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Answers

iii. Any scalar multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

iv. Show

v. $(3, 2)$

vi. Hint: use $ax_1 + by_1 + c = 0$

vii. Show. Insufficient to state the perpendicular distance formula.

Example 43

[2024 Sydney Girls HS Ext 1 Trial Q9] Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u} \cdot \mathbf{v} \neq 0$.

What does the following expression simplify to? **Answer:** (D)

$$\frac{\mathbf{u} \cdot \mathbf{v}}{(\mathrm{proj}_{\mathbf{v}}\, \mathbf{u}) \cdot (\mathrm{proj}_{\mathbf{u}}\, \mathbf{v})}$$

(A) $\sin^2\theta$ (B) $\operatorname{cosec}^2\theta$ (C) $\cos^2\theta$ (D) $\sec^2\theta$

Further exercises

Ex 3K (Haese et al., 2015)
(More introductory type questions)
- Every second subpart

Ex 8E (Pender et al., 2019)
- Q5-13

Part II

Vectors in three dimensions

Section 4

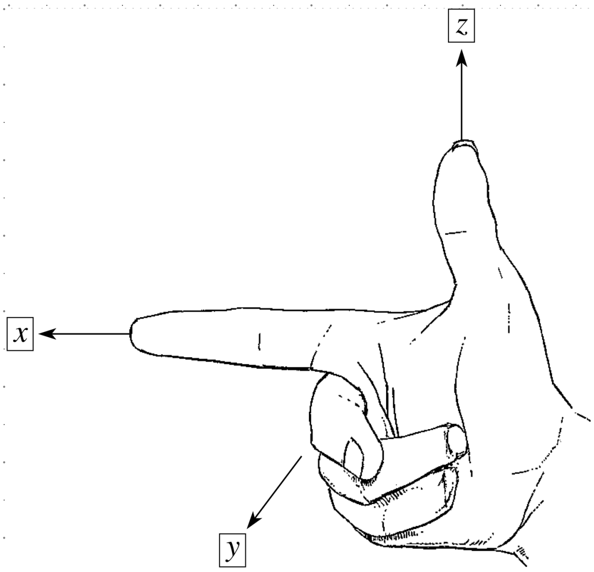
Cartesian coordinates in three dimensions

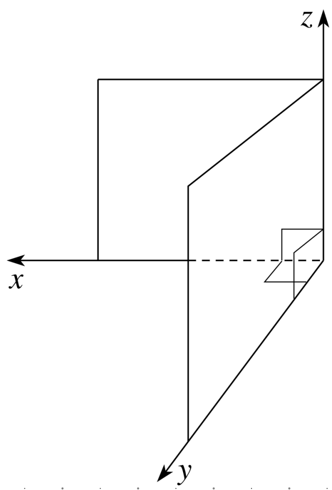
Fill in the spaces

- **The third axis**, called the .... , is required for drawing graphs in three dimensions.
- The .... is .... to both the x axis and y axis.
- The three axes in three dimensions are .... axes.

Important note

The **direction** of the ... . needs to be determined correctly!





Definition 18

Cartesian coordinates in two dimensions

- The two axes divide the plane into ....

Cartesian coordinates in three dimensions

- The xy-plane is the plane containing .... and ....
- The three planes, xy-plane, xz-plane and yz-plane, divide the 3D space into ....

Important note

- Later at university, 2D space will often be denoted $\mathbb{R}^2$ (pronounced "R two"), 3D space will be denoted $\mathbb{R}^3$ (pronounced "R three").
- Some references to $\mathbb{R}^2$ and $\mathbb{R}^3$ will be used throughout this summary.
- Scalars continue to be part of real numbers, e.g. $\lambda \in \mathbb{R}$. Vectors belonging to a particular space will be denoted

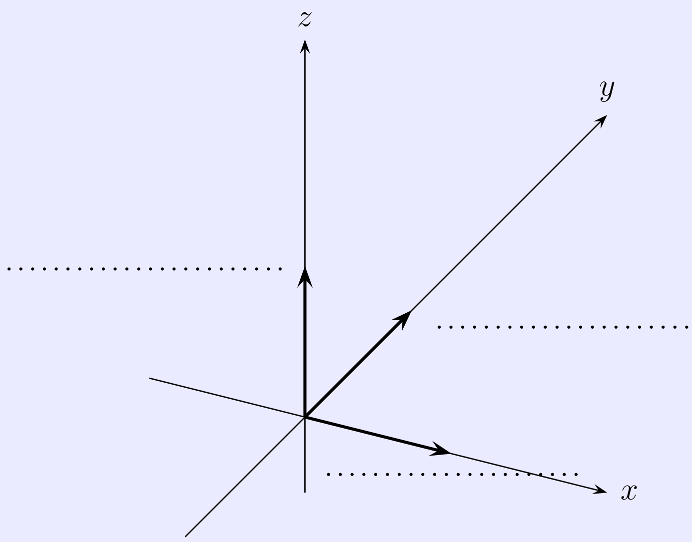
$$\mathbf{u} \in \mathbb{R}^2 \quad \text{or} \quad \mathbf{v} \in \mathbb{R}^3$$

as appropriate.

4.1 Vector components and operations

Definition 19

Basis vectors in 3D the .... **i**, **j** and **k** are aligned with the x, y and z axis respectively.



Definition 20

A **vector in three dimensions** can be expressed in a variety of forms:

- ....

$$\overrightarrow{OA} \quad \text{where} \quad A(a, b, c)$$

- .... form:

$$\overrightarrow{OA} = \mathbf{r} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$$

- .... notation:

$$\overrightarrow{OA} = \mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Definition 21

Dot product Suppose that $\mathbf{u} = \overrightarrow{OU} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\mathbf{v} = \overrightarrow{OV} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$. Then,

....

Section 5

Vector geometry

5.1 Parallel/perpendicular vectors

Laws/Results

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *parallel* if it can be shown that

$$\mathbf{a} = \ldots \mathbf{b}$$

for some $\lambda \in \mathbb{R}$.

Laws/Results

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *perpendicular* if it can be shown that

$$\ldots\ldots\ldots\ldots = 0$$

Example 44

[**2025 Sydney Grammar Ext 2 Trial Q5**] Consider the two vectors

$$\mathbf{u} = \begin{pmatrix} 3-t \\ t+1 \\ 2t-1 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} t+5 \\ 3t+3 \\ 3 \end{pmatrix}$$

For which value of t will the two vectors be parallel? **Answer:** (C)

(A) $t = -1$ (B) $t = 0$ (C) $t = 1$ (D) $t = 2$

Example 45

[2024 Ext 2 HSC Q12] The vector $\mathbf{a}$ is $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and the vector $\mathbf{b}$ is $\begin{pmatrix}2\\0\\-4\end{pmatrix}$.

i. Find $\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$. **1**

ii. Show that $\mathbf{a}-\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$ is perpendicular to $\mathbf{b}$. **2**

Answer: $\begin{pmatrix}-1\\0\\2\end{pmatrix}$

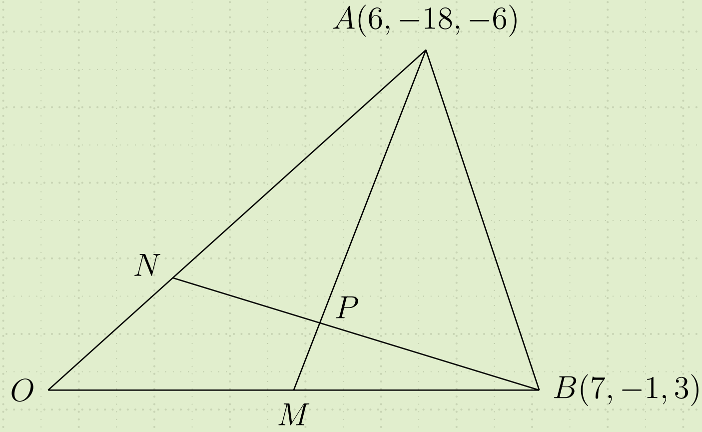
Example 46

[**2020 Ext 2 HSC Q11**] (3 marks) Consider the two vectors $\mathbf{u} = -2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{v} = p\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}$.

For what values of p are $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ perpendicular? **Answer:** $p = \pm 3$

Example 47

[2025 Baulkham Hills HS Ext 2 Trial Q13] (4 marks) The vertices of $\triangle OAB$ have coordinates $A(6,-18,-6)$ and $B(7,-1,3)$, and O is a fixed origin.



The point N lies on OA so that $ON : NA = 1 : 2$.

The point M is the midpoint of OB. The point P is the intersection of AM and BN.

By using vector methods, determine the coordinates of P. **Answer:** $P(4,-4,0)$

5.2 Magnitude

Definition 22

If $\mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then

$$|\mathbf{p}| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$|\mathbf{p}|^2 = \ldots\ldots\ldots\ldots$$

Example 48

[2025 CSSA Ext 2 Trial Q9] The vector $\mathbf{a}$ is given by $\sqrt{2}\,\mathbf{i} + \left(\dfrac{1}{\tan\theta}\right)\mathbf{j} - \tan\theta\,\mathbf{k}$, where $0 < \theta < 90°$.

Which of the following is an expression for $|\mathbf{a}|$? **Answer:** (A)

(A) $\sec\theta\operatorname{cosec}\theta$ (B) $\cot\theta$ (C) $\operatorname{cosec}\theta$ (D) 1

Example 49

[2024 JRAHS Ext 2 Trial Q12] If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$, find the value of

$$\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}$$

Answer: $-\frac{3}{2}$

Important note

Try to find an expansion for $|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2$

Example 50

[**2024 JRAHS Ext 2 Trial Q13**] Consider the two perpendicular vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

i. Find the unit vectors $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$. **2**

ii. Find a unit vector $\widehat{\mathbf{w}}$ that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. **2**

iii. Let the vector $\mathbf{x} = a\widehat{\mathbf{u}} + b\widehat{\mathbf{v}} + c\widehat{\mathbf{w}}$, where a, b and $c \in \mathbb{R}$. If $|\mathbf{x}| = 1$, prove that **2**

$$a^2 + b^2 + c^2 = 1$$

iv. Let **2**

- α be the measure of the angles between $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{x}}$
- β be the measure of the angles between $\widehat{\mathbf{v}}$ and $\widehat{\mathbf{x}}$
- γ be the measure of the angles between $\widehat{\mathbf{w}}$ and $\widehat{\mathbf{x}}$

Prove that

$$\cos\alpha + \cos\beta + \cos\gamma = a + b + c$$

Important note

Hint: the note from Example 49 will be handy!

5.3 Midpoint

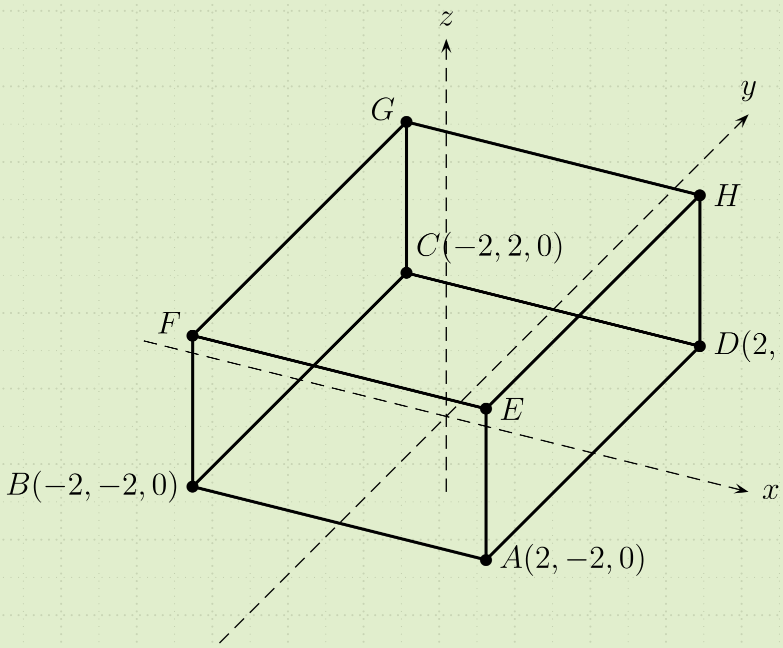
Definition 23

For coordinates $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then the midpoint M of PQ will have coordinates

$$M\left(\ldots\ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots\ldots\right)$$

Example 51

[2025 Penrith HS Ext 2 Trial Q15] For the rectangular prism shown:



i. Show that the internal diagonals AG and DG bisect each other. **2**

ii. Calculate the acute angle, θ, between the diagonals, correct to the nearest degree. **2**

Answer: $84°$

Further exercises

Ex 5A (Sadler & Ward, 2019)

- Q1-12

5.4 Angle between two vectors

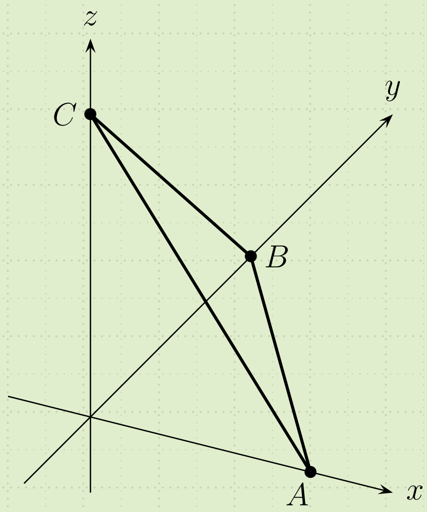
Laws/Results

Angle between two vectors The angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ can be found using

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

Example 52

[2023 Sydney Grammar Ext 2 Trial Q12] Let $\triangle ABC$ have vertices $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,k)$ for some constant $k>0$, as shown below.



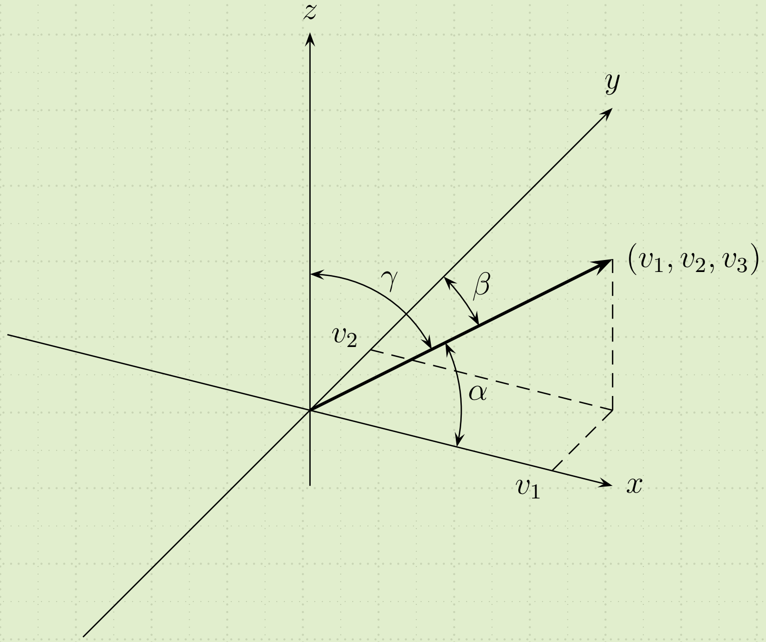
i. Find $\overrightarrow{AC}$ and $\overrightarrow{BC}$ as column vectors. **1**

ii. (Modified) If $\angle ACB = 45°$, show that **2**

$$k = \sqrt{\frac{1}{\sqrt{2}-1}}$$

Example 53

[2024 CSSA Ext 2 Trial Q13] The vector **v** is represented by the point with coordinates (v_1, v_2, v_3) as shown in the diagram.



Let α, β and γ be the angles between **v** and the positive x, y and z axes, respectively.

The direction cosines of a vector are the cosines of the angles between the vector and three positive coordinate axes. That is, the direction cosines of **v** are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$.

Answer: 30°

i. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ **2**

ii. A vector **r** makes an angle of 72° with the positive x axis and 36° with the positive y axis **1**

Find the angle of elevation of **r** from the xy plane.

Example 54

[2025 NSBHS Ext 2 Trial Q11] (3 marks) Points A, B and C are respectively defined by position vectors

$$\begin{aligned} \mathbf{a} &= \mathbf{i} - \mathbf{j} + 2\,\mathbf{k} \\ \mathbf{b} &= 2\,\mathbf{i} - \mathbf{j} + \mathbf{k} \\ \mathbf{c} &= p\,\mathbf{i} - 2\,\mathbf{j} \end{aligned}$$

Given that $\angle ABC = 60°$, find the value(s) of p. **Answer:** $p = -2$

Important note

Watch for unsuspecting restrictions on the values that p can take!

Further exercises

Ex 5C (Sadler & Ward, 2019)

- Q9, 11

5.5 Projection of a vector on to another vector

Important note

The projection formula is identical for 2D and 3D situations.

Example 55

[2025 Independent Ext 2 Trial Q11] Consider the vectors $\mathbf{u} = 2\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j} + 8\,\mathbf{k}$ and $\mathbf{v} = 6\,\mathbf{i} + 6\,\mathbf{j} + 2\sqrt{2}\,\mathbf{k}$.

i. Find the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. **2**

ii. Find the projection of the vector $\mathbf{u}$ on to the vector $\mathbf{v}$. **2**

Answer: i. $45°$ ii. $3\sqrt{2}\,\mathbf{i} + 3\sqrt{2}\,\mathbf{j} + 2\,\mathbf{k}$

Example 56

[2025 Sydney Grammar Ext 2 Trial Q13] Consider the points $A(0,-1,6)$, $B(2,3,5)$ and $C(1,4,4)$. Let $\mathbf{a} = \overrightarrow{CA}$ and $\mathbf{b} = \overrightarrow{CB}$. **Answer:** $3\sqrt{2}$

i. Show that $|\text{proj}_{\mathbf{b}}\,\mathbf{a}| = 2\sqrt{3}$. **2**

ii. Hence determine the minimum distance from A to the line BC. **1**

Timed exam practice 5 (Allow approximately 7 minutes)

[2025 Ext 2 HSC Q15] (4 marks) The adjacent sides of a parallelogram are represented by the vectors

$$\mathbf{a} = 4\,\mathbf{i} + 3\,\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{b} = 2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$$

Show that the area of the parallelogram is $6\sqrt{10}$ square units.

Marking criteria

✓ [1] Finds $|\mathbf{a}|$ or $|\mathbf{b}|$, or equivalent merit.

✓ [2] Calculates a projection, or equivalent merit.

✓ [3] Calculates the perpendicular height, or equivalent merit.

✓ [4] Provides correct solution.

Example 57

[2025 JRAHS Ext 2 Trial Q10] Let $\mathbf{u} = 2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$ and suppose $\mathbf{v}$ is a vector in three dimensions.

Let $\mathbf{p}$ be the vector projection of $\mathbf{u}$ on to $\mathbf{v}$, and let $\mathbf{n}$ be the vector component of $\mathbf{u}$ that is perpendicular to $\mathbf{v}$, that is, $\mathbf{u} = \mathbf{p} + \mathbf{n}$.

If $|\mathbf{p}| > |\mathbf{n}|$, which of the following could be $\mathbf{v}$? **Answer:** (B)

(A) $\mathbf{i} + \mathbf{j} + \mathbf{k}$

(B) $3\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$

(C) $\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$

(D) $3\,\mathbf{i} + 4\,\mathbf{j} + 2\,\mathbf{k}$

Further exercises

Ex 5D (Sadler & Ward, 2019)

- Q8-12

Additional questions

1. **[2025 Sydney Girls HS Ext 2 Trial Q4]** Suppose $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$, $|a|=3$, $|b|=5$ and $|c|=7$. **1**

What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

(A) $30°$ (B) $120°$ (C) $300°$ (D) $60°$

2. **[2025 NSBHS Ext 2 Trial Q5]** For the vectors **1**

$$\mathbf{p}=\begin{pmatrix}a\\b\\c\end{pmatrix} \quad \text{and} \quad \mathbf{q}=\begin{pmatrix}b\\c\\a\end{pmatrix}$$

which of the following is the expression for

$$\left|\mathrm{proj}_{\mathbf{q}}\,\mathbf{p}\right|$$

(A) $a^2+b^2+c^2$

(B) $|ab+bc+ca|$

(C) $\dfrac{a^2+b^2+c^2}{|ab+bc+ca|}$

(D) $\dfrac{|ab+bc+ca|}{\sqrt{a^2+b^2+c^2}}$

3. **[2025 NSBHS Ext 2 Trial Q9]** Points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively. **1**

Which of the following correctly describes this expression?

$$\frac{1}{2}\sqrt{|\mathbf{p}|^2\,|\mathbf{q}|^2-(\mathbf{p}\cdot\mathbf{q})^2}$$

(A) The perimeter of $\triangle OPQ$

(B) The area of $\triangle OPQ$

(C) The magnitude of the position vector of the midpoint of PQ

(D) The cosine of $\angle POQ$

4. **[2025 Independent Ext 2 Trial Q2]** $OABC$ is a square with $\overrightarrow{OA}=a\,\mathbf{i}-2\,\mathbf{j}-2\,\mathbf{k}$ and $\overrightarrow{OC}=6\,\mathbf{i}-4\,\mathbf{j}+a\,\mathbf{k}$ for some constant a. What is the value of a? **1**

(A) -2 (B) 1 (C) 1 (D) 2

5. **[2024 Sydney Grammar Ext 2 Trial Q5]** Let $\overrightarrow{OP}=\frac{1}{2}\left(\sqrt{2}\,\mathbf{i}-\mathbf{j}+\mathbf{k}\right)$, and α, β and γ be the angles that $\overrightarrow{OP}$ makes with the positive x, y and z axes respectively. **1**

What is the value of $\alpha+\beta+\gamma$?

(A) $45°$ (B) $165°$ (C) $180°$ (D) $225°$

6. **[2025 Hornsby Girls HS Ext 2 Trial Q12]** The vertices of $\triangle ABC$ are defined by the position vectors

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ -6 \\ -2 \end{pmatrix} \qquad \overrightarrow{OB} = \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} \qquad \overrightarrow{OC} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$$

i. Show that $\cos\angle BAC = \frac{1}{2}$. **3**

ii. Hence, find the exact area of $\triangle ABC$. **1**

7. **[2023 Baulkham Hills HS Ext 2 Trial Q14]** Find a unit vector that is perpendicular to both $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$. **3**

8. **[2024 Fort St HS Ext 2 Trial Q11]** The position vectors of two points , A and B are given by

$$\overrightarrow{OA} = 2\,\mathbf{i} + 4\,\mathbf{j} - 3\,\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = -\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}$$

i. Determine the exact distance between the points A and B. **1**

ii. Show that there is no value of m such that $\overrightarrow{OC} = m\,\mathbf{i} + 2\,\mathbf{j} - m^2\,\mathbf{k}$ is perpendicular to $\overrightarrow{OA}$. **2**

9. **[2024 Killara HS Ext 2 Trial Q13]** Let $\mathbf{u} = \mathbf{i} + \mathbf{j} + z\,\mathbf{k}$ and $\mathbf{v} = 2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$. **3**

Find all z such that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $60°$.

10. Let $\mathbf{a} = 3\,\mathbf{i} + 2\,\mathbf{j} + m\,\mathbf{k}$ and $\mathbf{b} = 4\,\mathbf{i} - \mathbf{j} + m^2\,\mathbf{k}$, where m is a real constant. It is given that the scalar projection of $\mathbf{a}$ in the direction of $\mathbf{b}$ is $\dfrac{74}{\sqrt{273}}$. **3**

Show that the value of m is 4.

11. ⚠ **[2024 Sydney Girls HS Ext 2 Trial Q10]** Suppose there exists vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}$ such that $\mathbf{r} = \mathbf{a} + (\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}$. **1**

If $\mathbf{r}$ is perpendicular to $\mathbf{a} - \mathbf{b}$, what is the minimum value of $1 + \mathbf{b}^2$?

(A) $|\mathbf{a}| + 1$ (B) $\dfrac{|\mathbf{a}| + 1}{2}$ (C) $2\,|\mathbf{a}|$ (D) $|\mathbf{a}|$

Answers

1. (D) **2.** (D) **3.** (B) **4.** (A) **5.** (D) **6.** $3\sqrt{3}$ **7.** $\frac{1}{3\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ -7 \end{pmatrix}$ **8.** i. $\sqrt{43}$ ii. Show **9.** $\frac{-6+2\sqrt{42}}{11}$ **10.** Show **11.** (C)

Section A

Past examination questions

A.1 2020 Ext 1 HSC

4. Maria starts at the origin and walks along all of the vector $2\,\mathbf{i}+3\,\mathbf{j}$, then walks along all of the vector $3\,\mathbf{i}-2\,\mathbf{j}$ and finally along all of the vector $4\,\mathbf{i}-3\,\mathbf{j}$. **1**

How far from the origin is she?

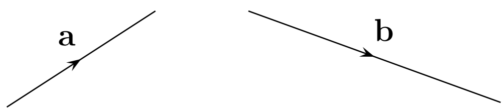
(A) $\sqrt{77}$

(B) $\sqrt{85}$

(C) $2\sqrt{13}+\sqrt{5}$

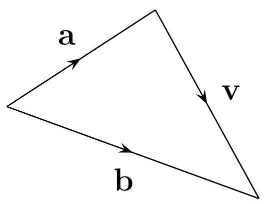
(D) $\sqrt{5}+\sqrt{7}+\sqrt{13}$

6. The vectors $\mathbf{a}$ and $\mathbf{b}$ are shown. **1**

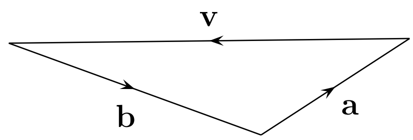


Which diagram below shows the vector $\mathbf{v} = \mathbf{a} - \mathbf{b}$?

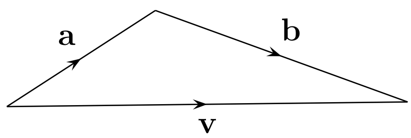
(A)



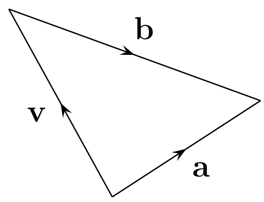
(C)



(B)



(D)



9. The projection of the vector $\begin{pmatrix} 6 \\ 7 \end{pmatrix}$ on to the line $y = 2x$ is $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$. **1**

The point $(6, 7)$ is reflected in the line $y = 2x$ to a point A.

What is the position vector of the point A?

(A) $\begin{pmatrix} 6 \\ 12 \end{pmatrix}$ (B) $\begin{pmatrix} 2 \\ 9 \end{pmatrix}$ (C) $\begin{pmatrix} -6 \\ 7 \end{pmatrix}$ (D) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Question 11

(b) For what value(s) of a are the vectors $\begin{pmatrix} a \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2a-3 \\ 2 \end{pmatrix}$ perpendicular? **3**

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8. The angle between two unit vectors $\mathbf{a}$ and $\mathbf{b}$ is θ and $|\mathbf{a} + \mathbf{b}| < 1$. **1**

Which of the following best describes the possible range of values of θ?

(A) $0 \le \theta < \dfrac{\pi}{3}$

(B) $0 \le \theta < \dfrac{2\pi}{3}$

(C) $\dfrac{\pi}{3} < \theta \le \pi$

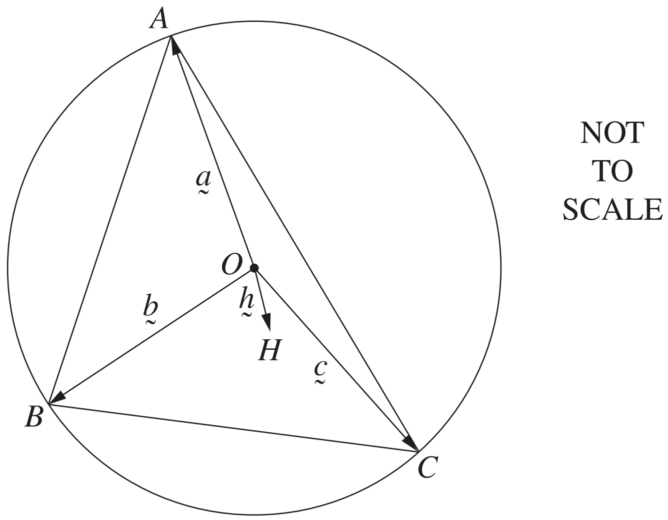
(D) $\dfrac{2\pi}{3} < \theta \le \pi$

Note: attempt this after Topic 9 (Radians) has been completed.

Question 13

(a) Three different points A, B and C are chosen on a circle centred at O. **3**

Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. Let $\mathbf{h} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ and let H be the point such that $\overrightarrow{OH} = \mathbf{h}$, as shown in the diagram.



Show that $\overrightarrow{BH}$ and $\overrightarrow{CA}$ are perpendicular.

Question 14

(b) The vectors $\mathbf{u}$ and $\mathbf{v}$ are not parallel. The vector $\mathbf{p}$ is the projection of $\mathbf{u}$ onto the vector $\mathbf{v}$. **3**

The vector $\mathbf{p}$ is parallel to $\mathbf{v}$ so it can be written as $\lambda_0\mathbf{v}$ for some real number λ_0. (Do NOT prove this).

Prove that $|\mathbf{u} - \lambda\mathbf{v}|$ is smallest when $\lambda = \lambda_0$ by showing that, for all real numbers λ, $|\mathbf{u} - \lambda_0\mathbf{v}| \leq |\mathbf{u} - \lambda\mathbf{v}|$.

Note: this question is likely to have moved into Extension 2 for the 2027 HSC.

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Question 14

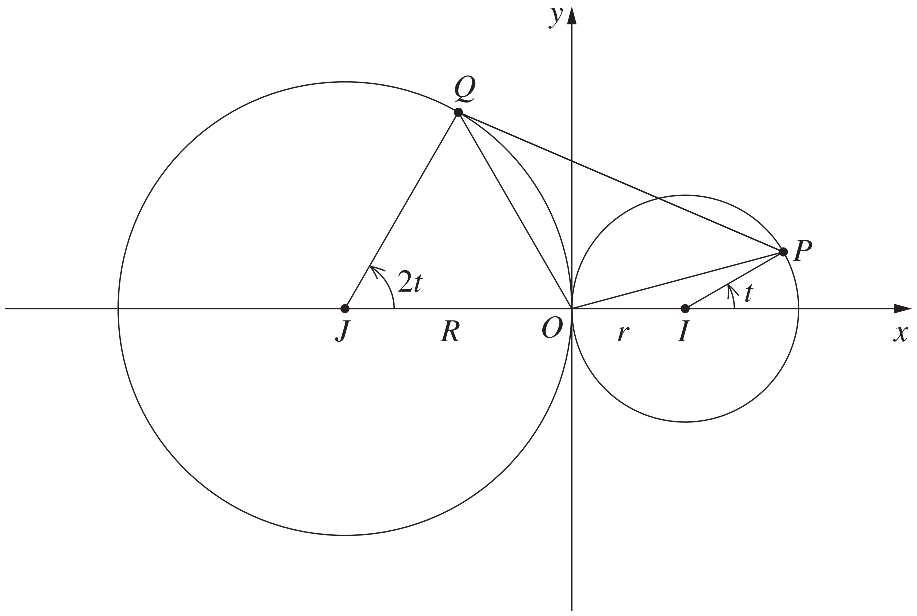
(c) i. Given a non-zero vector $\begin{pmatrix} p \\ q \end{pmatrix}$, it is known that the vector $\begin{pmatrix} q \\ -p \end{pmatrix}$ is perpendicular to $\begin{pmatrix} p \\ q \end{pmatrix}$ and has the same magnitude. (Do NOT prove this). **3**

Points A and B have position vectors $\overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, respectively.

Using the given information, or otherwise, show that the area of $\triangle OAB$ is $\dfrac{1}{2}|a_1b_2 - a_2b_1|$.

ii. The point P lies on the circle centred at $I(r, 0)$ with radius $r > 0$, such that $\overrightarrow{IP}$ makes an angle of t to the horizontal. **4**

The point Q lies on the circle centred at $J(-R, 0)$ with radius $2t$ to the horizontal.



Note that $\overrightarrow{OP} = \overrightarrow{OI} + \overrightarrow{IP}$ and $\overrightarrow{OQ} = \overrightarrow{OJ} + \overrightarrow{JQ}$.

Using part (i), or otherwise, find the values of t, where $-\pi \leq t \leq \pi$, that maximise the area of $\triangle OPQ$.
Note: attempt this after Topic 23 (Calculus with Trigonometric Functions) has been completed.

NESA Reference Sheet – Mathematics Extension 1 (includes Advanced)

Mathematics Extension 1

Trigonometric functions

Compound angles

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Functions

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\dfrac{d}{a}$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$

$X \sim \mathrm{Bin}(n, p)$

$\mathrm{E}(X) = np$

$\mathrm{Var}(X) = np(1-p)$

Differential calculus

Function	Derivative
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1 - [f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1 - [f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1 + [f(x)]^2}$

Integral calculus

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{a}{a^2 + x^2}\, dx = \tan^{-1}\left(\frac{x}{a}\right) + c$$

Vectors

$|\mathbf{u}| = |x\mathbf{i} + y\mathbf{j}| = \sqrt{x^2 + y^2}$

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = x_1x_2 + y_1y_2$

where $\mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j}$

and $\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j}$

$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \left(\mathbf{a} \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}}$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \cdots + {}^nC_{n-1} x y^{n-1} + {}^nC_n y^n$$

Mathematics Advanced

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}\left(a + b\right)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Financial mathematics

$$FV = PV\left(1 + r\right)^n$$

Sequences and series

$$a_n = a + \left(n - 1\right)d$$

$$S_n = \frac{n}{2}\left[2a + \left(n - 1\right)d\right] = \frac{n}{2}\left(a + a_n\right)$$

$$a_n = ar^{n-1}$$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}, \ r \neq 1$$

$$S_\infty = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and exponential functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Relations

$$\left(x - a\right)^2 + \left(y - b\right)^2 = r^2$$

Trigonometric functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

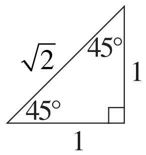
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

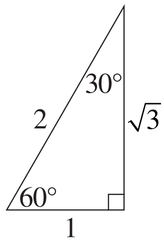
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Differential calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^x$	$\frac{dy}{dx} = (\ln a)a^x$
$y = \log_a x$	$\frac{dy}{dx} = \frac{1}{x\ln a}$

Integral calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int_a^b f(x)\,dx$$

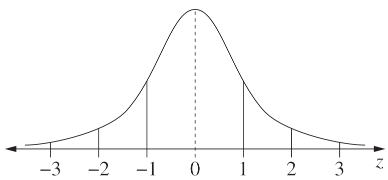
$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Statistical analysis

$$z = \frac{x - \mu}{\sigma}$$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

Discrete random variables

$$\mathrm{E}(X) = \mu$$

$$\mathrm{Var}(X) = \mathrm{E}\left[(X-\mu)^2\right]$$

Continuous random variables

$$P(X \leq x) = F(x) = \int_a^x f(t)\,dt$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

$$\mathrm{E}(X) = \mu = \int_a^b xf(x)\,dx$$

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \mu^2 = \int_a^b x^2 f(x)\,dx - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0$$

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