Inverse Trig Without Triangles

For the HSC Mathematics Extension 1/2 Student

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v2

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1 Introduction

This document provides a rigorous, **algebra and identities** based approach to HSC Mathematics Extension 1 inverse trig, moving beyond the imprecise "right-angled triangle" method taught in most schools. The first section contains and proves all the fundamental inverse trig identities, and the second section invovles more complex/niche identities. A variety of handcrafted practise problems and a cheat-sheet style summary are included at the end.

Why Is Right-Angled Triangle Bad?

The right-angled triangle is a useful mnemonic, but no more than a mnemonic, as it is imprecise. It should never appear in proofs involving inverse trig, for the following reasons:

- The angles in a right-angled triangle range from 0 to 90 degrees; a mere subset of the range of the inverse trig functions.
- The sides of a triangle are always positive, but the coordinates that define trig ratios can be negative. The triangle forces us to first ignore the signs, then compensate for them using ASTC. This is the main point of imprecision.
- We lose the notion of the inverse trig functions being actual functions!
- For more complex tasks, we may need to exhaust many tedious sign-related cases. By using identities, we often don't need cases at all.

Preliminaries

It is assumed that the reader has a solid understanding of regular trig, and is familiar with the inverse trig functions. The following table table provides a recap.

Function Name	Domain	Range	Inverse Property
$\sin^{-1}$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\sin(\sin^{-1} x) = x$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos(\cos^{-1} x) = x$
$\tan^{-1}$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\tan(\tan^{-1} x) = x$

Note: The inverse propoerties shown above are valid for the given domains. For example, $\sin(\sin^{-1} x) = x$ is true for all $x \in [-1, 1]$. However, the reverse, $\sin^{-1}(\sin x) = x$, is only true for a restricted domain, namely $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The inverse propoerties shown in the table are sometimes referred to as the "right inverse" properties because the inverse function is applied to the right of the original function. One must be mindful with domain restrictions when applying the inverse trig functions to the left of the original function.

2 Fundamental Inverse Trig Identities

In this section, we explore and prove five fundamental inverse trig identities that the Extension 1 student should be familiar with and can use/prove on demand. We start with a simple, yet useful theorem.

Theorem 2.1. *The inverse of an odd function is odd.*

Proof. Suppose f is an odd function with inverse f^{-1}. For any $y = f(x)$ in the domain of f^{-1},

$$\begin{aligned} f^{-1}(-y) &= f^{-1}(-f(x)) \\ &= f^{-1}(f(-x)) && \text{(as } f \text{ is odd)} \\ &= -x \\ &= -f^{-1}(y). \end{aligned}$$

Thus, f^{-1} is odd. □

Theorem 2.2. $\sin^{-1}(-x) = -\sin^{-1} x$.

Proof. As $\sin^{-1}$ is the inverse of the odd function $\sin x$ restricted to $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, by Theorem 2.1, $\sin^{-1}$ is odd. Hence, by the definition of an odd function, $\sin^{-1}(-x) = -\sin^{-1} x$. □

Theorem 2.3. $\tan^{-1}(-x) = -\tan^{-1} x$.

Proof. Left as an exercise. The proof is similar to the proof of Theorem 2.2. □

Theorem 2.4 (Complementary Angle Identity). $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

Proof. Let $x = \cos y$ for any $y \in [0, \pi]$. Since we know that $\cos y = \sin\left(\frac{\pi}{2} - y\right)$, we have

$$x = \sin\left(\frac{\pi}{2} - y\right).$$

Now, as $\frac{\pi}{2} - y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which is the range of $\sin^{-1}$, applying $\sin^{-1}$ to both sides of the above equation yields

$$\sin^{-1} x = \frac{\pi}{2} - y.$$

But $y = \cos^{-1} x$, so upon substitution, we complete the proof. □

Theorem 2.5. $\cos^{-1} x + \cos^{-1}(-x) = \pi$.

Proof. Applying Theorem 2.4 to both terms on the left hand side,

$$\begin{aligned} \cos^{-1} x + \cos^{-1}(-x) &= \frac{\pi}{2} - \sin^{-1} x + \frac{\pi}{2} - \sin^{-1}(-x) \\ &= \pi - \sin^{-1} x - (-\sin^{-1} x) && \text{(by Theorem 2.2)} \\ &= \pi. \end{aligned}$$

□

Remark. Theorem 2.5 can be rearranged as

$$\cos^{-1}(-x) - \frac{\pi}{2} = -\left(\cos^{-1} x - \frac{\pi}{2}\right),$$

which verifies that the graph $y = \cos^{-1} x$ is odd about the point $(x, y) = \left(0, \frac{\pi}{2}\right)$.

Theorem 2.6. $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \begin{cases} \pi/2 & \text{if } x > 0 \\ -\pi/2 & \text{if } x < 0 \end{cases}$

Proof. Define the function f by the rule

$$f(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x}$$

for $x \neq 0$. As f is differentiable over its domain and its derivative f' is given by

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{x^2} \cdot \frac{x^2}{x^2+1} = 0,$$

the function f must be piecewise-constant on all intervals it is differentiable. That is,

$$f(x) = \begin{cases} c_1 & \text{if } x > 0 \\ c_2 & \text{if } x < 0 \end{cases}$$

for some real constants c_1 and c_2. By substituting values $x = 1$ and $x = -1$, we find that $c_1 = \frac{\pi}{2}$ and $c_2 = -\frac{\pi}{2}$, thus proving our claim. □

3 Further Inverse Trig Identities

In this section, we explore and prove six further inverse trig identities that the Extension 1 student would find useful to recall and prove on demand. At the end of this section, we go through a few examples of exam-style questions, solving them elegantly by using these identities.

Theorem 3.1. $\cos(\sin^{-1} x) = \sqrt{1-x^2}$.

Proof. By the Pythagorean Identity,

$$\begin{aligned} \cos^2(\sin^{-1} x) &= 1 - \sin^2(\sin^{-1} x) \\ &= 1 - x^2 \\ \implies |\cos(\sin^{-1} x)| &= \sqrt{1-x^2}. \end{aligned}$$

But $\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and cos is non-negative for all values in this interval. Thus, we may drop the absolute value bars to obtain the result

$$\cos(\sin^{-1} x) = \sqrt{1-x^2}.$$

□

Remark. By setting $\theta = \sin^{-1} t$, the point $(\cos\theta, \sin\theta)$ on the unit circle becomes

$$\left(\cos(\sin^{-1} t), \sin(\sin^{-1} t)\right) = \left(\sqrt{1-t^2}, t\right).$$

Thus the identity $\cos(\sin^{-1} t) = \sqrt{1-t^2}$ can be interpreted geometrically as the positive x-coordinate of the point on the unit circle whose y-coordinate is t. Equivalently, the map

$$t \in [-1, 1] \mapsto \left(\sqrt{1-t^2}, t\right)$$

parameterises the right half of the unit circle.

Theorem 3.2. $\sin(\cos^{-1} x) = \sqrt{1-x^2}$.

Proof. Left as an exercise. The proof is similar to the proof of Theorem 3.1. □

Theorem 3.3. $\sec(\tan^{-1} x) = \sqrt{1+x^2}$.

Proof. Left as an exercise. Hint: use the identity $\sec^2\theta = 1 + \tan^2\theta$. □

Theorem 3.4. $\sin(\tan^{-1} x) = \dfrac{x}{\sqrt{1+x^2}}$.

Proof. We begin by writing the identity $\tan(\tan^{-1} x) = x$ as

$$\frac{\sin(\tan^{-1} x)}{\cos(\tan^{-1} x)} = x,$$

then upon dividing both sides by $\sec(\tan^{-1} x) = \sqrt{1+x^2}$ (Theorem 3.3), we obtain

$$\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}.$$

□

The following lemma will be useful to prove the next identity.

Lemma 3.5. *If* $xy < 1$, *then* $-\frac{\pi}{2} < \tan^{-1} x + \tan^{-1} y < \frac{\pi}{2}$.

Proof. Let $\alpha = \tan^{-1} x$ and $\beta = \tan^{-1} y$. Since $-\frac{\pi}{2} < \alpha, \beta < \frac{\pi}{2}$, we must have

$$-\pi < \alpha + \beta < \pi.$$

Our goal now is to tighten this inequality. By the cos compound angle identity,

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
&= \frac{1 - xy}{\sqrt{1+x^2}\sqrt{1+y^2}} && \text{(by Theorems 3.3 and 3.4)} \\
&> 0. && \text{(as } xy < 1\text{)}
\end{aligned}$$

The subset of $(-\pi, \pi)$ where cos is positive is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so it follows that

$$-\frac{\pi}{2} < \alpha + \beta < \frac{\pi}{2}.$$

□

Theorem 3.6 (Arctangent Addition Formula)**.** *If* $xy < 1$, *then*

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right).$$

Proof. Let $\alpha = \tan^{-1} x$ and $\beta = \tan^{-1} y$, where $xy < 1$. By the tan compound angle identity,

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{x+y}{1-xy}.$$

By Lemma 3.5, $\alpha + \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ which is the range of $\tan^{-1}$, so by applying $\tan^{-1}$ to both sides, we arrive at the result

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right).$$

□

Theorem 3.7 (Arctangent Subtraction Formula)**.** *If* $xy > -1$, *then*

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right).$$

Proof. Replace y with $-y$ in Theorem 3.6. □

Example 3.1. Prove the identity

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}.$$

Solution. We begin by writing $\tan(\sin^{-1} x) = \dfrac{\sin(\sin^{-1} x)}{\cos(\sin^{-1} x)}$. Since $\sin(\sin^{-1} x) = x$, and by Theorem 3.1 $\cos(\sin^{-1} x) = \sqrt{1-x^2}$, we swiftly arrive at the result

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}.$$

□

Example 3.2. Solve the equation

$$\sin^{-1}(2x) = \tan^{-1}(5x).$$

Solution. As both sides of the equation are in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and sin is one-to-one on this interval, we may apply sin to both sides. By Theorem 3.4, we obtain the equivalent equation,

$$2x = \frac{5x}{\sqrt{1+25x^2}}.$$

Clearly $x = 0$ is a solution. In the case where $x \neq 0$, squaring both sides and simplifying yields the quadratic equation

$$100x^2 - 21 = 0,$$

and so our solutions are $x = 0, \pm\dfrac{\sqrt{21}}{10}$. □

Example 3.3 (James Ruse '22 14c)**.** Show that for all real $-1 < x < 1$,

$$\cos\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) + \cos^{-1} x\right) = 0.$$

Solution. By expanding the left hand side with the cos compound angle identity and applying Theorems 3.2, 3.3 and 3.4,

$$\begin{aligned} LHS &= \cos\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)\cos(\cos^{-1} x) - \sin\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)\sin(\cos^{-1} x) \\ &= \frac{x}{\sqrt{1+\frac{x^2}{1-x^2}}} - \frac{\frac{x}{\sqrt{1-x^2}}\cdot\sqrt{1-x^2}}{\sqrt{1+\frac{x^2}{1-x^2}}} \\ &= 0. \end{aligned}$$

Nice and easy! □

Example 3.4 (Gosford '20 14d)**.** Find the six solutions of the equation

$$\sin\left(2\cos^{-1}\left(\cot\left(2\tan^{1} x\right)\right)\right) = 0.$$

Solution. Our strategy is to simplify the left hand side expression from the inside-out. By the tan double angle identity, we have

$$\tan(2\tan^{-1} x) = \frac{2x}{1-x^2},$$

implying $\cot(2\tan^{-1} x) = \dfrac{1-x^2}{2x}$. Then using the sin double angle identity, the equation becomes

$$2\sin\left(\cos^{-1}\left(\frac{1-x^2}{2x}\right)\right)\cos\left(\cos^{-1}\left(\frac{1-x^2}{2x}\right)\right) = 0,$$

which by Theorem 3.2 simplifies to

$$\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}\cdot\frac{1-x^2}{x} = 0.$$

Now, by the null factor law, we have three cases to consider.

Case 1. $1 - x^2 = 0$, which has solutions $x = \pm 1$.

Case 2. $\dfrac{1-x^2}{2x} = 1$, which rearranges to $x^2 + 2x - 1 = 0$, which has solutions $x = -1 \pm \sqrt{2}$.

Case 3. $\dfrac{1-x^2}{2x} = -1$, which rearranrges to $x^2 - 2x - 1 = 0$, which has solutions $x = 1 \pm \sqrt{2}$.

Thus, the solutions to the equation are $x = \pm 1, -1 \pm \sqrt{2}, 1 \pm \sqrt{2}$. □

4 Problems

Below are handcrafted problems for the kind reader! The identities throughout this document will be useful.

Problem 1. Solve the equation
$$3\sin^{-1}x = 5\cos^{-1}x.$$

Problem 2.

(i) Prove the identity $\tan\left(\cos^{-1}\dfrac{1}{x}\right) = \dfrac{x\sqrt{x^2-1}}{|x|}$.

(ii) Hence, sketch the graph of $y = \tan\left(\cos^{-1}\dfrac{1}{1+x}\right)$.

Problem 3.

(i) Show that $\tan\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{\sin\theta}$.

(ii) Deduce that $\tan\left(\dfrac{1}{2}\sin^{-1}x\right) = \dfrac{1-\sqrt{1-x^2}}{x}$ for $x \neq 0$.

Problem 4. Solve the equation
$$\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}.$$

Problem 5. Prove the identity $\sin^{-1}\sqrt{1-x^2} + \left|\sin^{-1}x\right| = \dfrac{\pi}{2}$.

Problem 6. Solve the equation
$$\cos^{-1}x + \cos^{-1}(x^2) = \frac{\pi}{2}.$$

Express your answer(s) in terms of φ, where $\varphi = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio.

Problem 7 (NSWMaths MX1 POTD #253)**.**

Define the function $f(x) = \left(\sin^{-1}x\right)^2 + \left(\cos^{-1}x\right)^2$.

(i) Show that $f'(x) = \dfrac{4\sin^{-1}x - \pi}{\sqrt{1-x^2}}$.

(ii) By considering the stationary points of f, determine the values of k for which the equation $f(x) = k$ has exactly two solutions, and find these solutions in terms of k.

Problem 8. Prove the identity $\cos^{-1}(\cos x) = 2\sin^{-1}\left|\sin\dfrac{x}{2}\right|$.

Problem 9. Define the function $f(x) = \cos\left(\tan^{-1}\left(\cos\left(2\sin^{-1}x\right)\right)\right)$.

(i) Show that $f(x) = \dfrac{1}{\sqrt{4x^4 - 4x^2 + 2}}$ for $-1 \leq x \leq 1$.

(ii) Hence, find the stationary points of $f(x)$ and determine their nature.

Problem 10. Suppose a, b, c are positive numbers such that $ab + bc + ca = 1$. Prove that

$$\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \frac{\pi}{2}.$$

Problem 11. Solve the equation

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \tan^{-1}(x+1) + \tan^{-1}(x-1).$$

Problem 12. Prove the identity

$$\tan^{-1}\left(\sqrt{1+t^2} - t\right) = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}t.$$

Problem 13. Define the function $f(x) = \cos^{-1}\left(\dfrac{1-x}{1+x}\right)$.

(i) Show that $f(x) = 2\tan^{-1}\sqrt{x}$.

(ii) A funnel is constructed by rotating the region bounded by the curve $y = f(x)$, the line $y = \dfrac{\pi}{2}$ and the y-axis, about the y-axis.

Find the exact volume of the funnel.

Problem 14. Suppose $\mathbf{u}$ and $\mathbf{v}$ are unit vectors such that

$$\sin^{-1}(\mathbf{u} \cdot \mathbf{v}) = 5\tan^{-1}\left(\frac{|\mathbf{u}-\mathbf{v}|}{|\mathbf{u}+\mathbf{v}|}\right).$$

What is the angle between $\mathbf{u}$ and $\mathbf{v}$?

Problem 15.

(i) Prove that for $x^2 + y^2 \leq 1$,

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right).$$

Hint: Consider a similar approach to the proof of Theorem 3.6.

(ii) Hence, or otherwise, prove that for $-\dfrac{1}{2} \leq x \leq \dfrac{1}{2}$,

$$3\sin^{-1}x = \sin^{-1}(3x - 4x^3).$$

Problem 16 (NSWMaths MX2 POTD #268)**.**

Define the function $\phi(x) = \dfrac{\sqrt{1+x^2}-1}{x}$ for $x \neq 0$.

Define $\phi \circ \phi(x)$ to be the composition of $\phi(x)$ with itself, that is, $\phi \circ \phi(x) = \phi(\phi(x))$.

Define $\phi^n(x)$ to be the n-th iteration of $\phi(x)$ for integers $n \geq 0$, that is,

$$\phi^n(x) = \begin{cases} x & \text{if } n = 0 \\ \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n\ \phi\text{'s}}(x) & \text{if } n \geq 1. \end{cases}$$

(i) Using the substitution $x = \tan\theta$ for a suitably chosen domain of θ, prove that for all $x \neq 0$,

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\tan^{-1} x.$$

(ii) **This part is Extension 2, but it should be ok for a strong Extension 1 student.**

Prove using induction that for all integers $n \geq 0$,

$$\phi^n(x) = \tan\left(2^{-n}\tan^{-1} x\right).$$

(iii) Hence, show that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\tan^{-1}\left(\phi^k(x)\right) = \tan^{-1} x.$$

(iv) Find the rule for the inverse function ϕ^{-1} using only the functions tan and $\tan^{-1}$, and state its domain and range.

(v) **Do not attempt this part unless you do competition maths.**

Find all continuous functions $f : (0,1) \to \mathbb{R}$ satisfying

$$f(x) = f(\phi(x)) + f(\phi^{-1}(x)).$$

Problem 17 (Requires Extension 2 Techniques)**.** Prove that the equation

$$\cos^{-1}\left(e^{-x}\right) = \tan^{-1} x$$

has no solutions for $x > 0$.

5 Cheat Sheet

Function Name	Domain	Range	Inverse Property
$\sin^{-1}$	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$	$\sin(\sin^{-1}x)=x$
$\cos^{-1}$	$[-1,1]$	$[0,\pi]$	$\cos(\cos^{-1}x)=x$
$\tan^{-1}$	$(-\infty,\infty)$	$(-\frac{\pi}{2},\frac{\pi}{2})$	$\tan(\tan^{-1}x)=x$

Fundamental Inverse Trig Identities

- $\sin^{-1}(-x) = -\sin^{-1}x$
- $\tan^{-1}(-x) - \tan^{-1}x$
- $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$
- $\cos^{-1}x + \cos^{-1}(-x) = \pi$
- $\tan^{-1}x + \tan^{-1}\dfrac{1}{x} = \begin{cases} \pi/2 & \text{if } x > 0 \\ -\pi/2 & \text{if } x < 0 \end{cases}$

Further Inverse Trig Identities

- $\cos(\sin^{-1}x) = \sqrt{1-x^2}$
- $\sin(\cos^{-1}x) = \sqrt{1-x^2}$
- $\sec(\tan^{-1}x) = \sqrt{1+x^2}$
- $\sin(\tan^{-1}x) = \dfrac{x}{\sqrt{1+x^2}}$
- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) \quad \text{for } xy < 1$
- $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) \quad \text{for } xy > -1$

6 The Out-of-Syllabus Functions and a Common Misbelief

The lesser known $\cot^{-1} x$, $\sec^{-1} x$ and $\operatorname{cosec}^{-1} x$ functions are outside the scope of the HSC Mathematics Extension 1 course. For completeness, it is useful to know their common definitions in terms of the standard inverse trig functions:

- $\cot^{-1} x = \dfrac{\pi}{2} - \tan^{-1} x$
- $\sec^{-1} x = \cos^{-1} \dfrac{1}{x}$
- $\operatorname{cosec}^{-1} x = \sin^{-1} \dfrac{1}{x}$

For the purposes of this document, we do not discuss these functions further.

A common misbelief amongst students is the statement,

$$\cot^{-1} x = \tan^{-1} \frac{1}{x} \quad \text{for all non-zero } x,$$

which is a natural assumption to make based on the similar definitions of $\sec^{-1}$ and $\operatorname{cosec}^{-1}$. This is in fact false, due to the nuanced definition of $\cot^{-1}$. This statement is valid if and only if $x > 0$.

7 Answers to Problems

Rotate your screen!

1. $\cos\left(\frac{3\pi}{16}\right)$
4. $x = \frac{1}{6}$
6. $x = \frac{1}{\sqrt{\varphi}}$
7. $\frac{\pi^2}{8} < k \leq \frac{\pi^2}{4}$, $x = \sin\left(\frac{\pi \pm \sqrt{8k-\pi^2}}{4}\right)$
9. (ii) local max at $x = \pm\frac{1}{\sqrt{2}}$, local min at $x = 0$
11. $x = 0, \pm\frac{\sqrt{6}}{2}$
13. (ii) $\frac{\pi^2}{2} - \frac{4\pi}{3}$
14. $\frac{\pi}{7}$
16. (iv) $\phi^{-1}(x) = \tan\left(2\tan^{-1} x\right)$, D: $(-1,0) \cup (0,1)$, R:$(-\infty,0) \cup (0,\infty)$

...Curious about the answers to the rest? Join NSWMaths!