2025 **MOCK HSC EXAM**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–11)

- Attempt Questions 11–14
- Allow about 1 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Consider the vector $\overrightarrow{OP} = 18\underset{\sim}{i} + 7\underset{\sim}{j}$ and the point $Q = (-34, -6)$.

The point which internally divides PQ in the ratio $6 : 7$ is:

A. $(6, 1)$ B. $(-6, -1)$ C. $(-6, 1)$ D. $(6, -1)$

2 The roots of a cubic polynomial are α, β, γ. Given that $\alpha + \beta + \gamma = 4$, $\alpha\beta + \beta\gamma + \gamma\alpha = -5$ and $\alpha\beta\gamma = 6$, which of the following is the monic cubic polynomial?

A. $x^3 - 4x^2 - 5x - 6$ B. $x^3 - 4x^2 - 5x + 6$

C. $x^3 - 4x^2 + 5x - 6$ D. $x^3 - 4x^2 - 5x + 6$

3 Twelve men line up in a line. Three of the men are discord moderators, and will complain if they have to stand next to another discord moderator. If the men line up at random, what is the probability a moderator complains?

A. $\frac{4}{11}$ B. $\frac{5}{11}$ C. $\frac{6}{11}$ D. $\frac{7}{11}$

4 The random variable X represents the number of successes of 10 Bernoulli trials with a probability of success of 0.4. Find $P(X < 3 | X > 1)$, correct to two decimal places.

A. 0.12 B. 0.13 C. 0.17 D. 0.18

5 Using the substitution $u = \sqrt{t}$, the value of $\displaystyle\int_x^{4x} \frac{1}{t + \sqrt{t}}\, dt$ is:

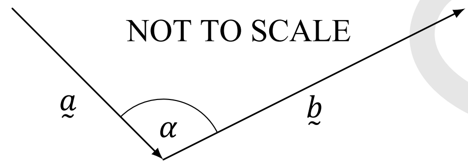
A. $2\ln\left(\dfrac{2\sqrt{x}+1}{\sqrt{x}+1}\right)$

B. $2\ln\left(\dfrac{\sqrt{x}+1}{2\sqrt{x}+1}\right)$

C. $\dfrac{1}{2}\ln\left(\dfrac{2\sqrt{x}+1}{\sqrt{x}+1}\right)$

D. $\dfrac{1}{2}\ln\left(\dfrac{\sqrt{x}+1}{2\sqrt{x}+1}\right)$

6 Consider the following diagram depicting 2D vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and the angle α.



It is known that $|\underset{\sim}{a}| = 187$, $|\underset{\sim}{b}| = 781$, and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 17871$. To the nearest degree, what is the size of angle α?

A. $97°$ B. $83°$ C. $84°$ D. $1°$

7 The graph of the function $y = 3\cos^{-1} x + \pi$ is transformed by being reflected about the y-axis, dilated vertically by a factor of $\frac{1}{2}$, and then translated down by $\frac{3\pi}{4}$ units.

What is the equation of the transformed graph?

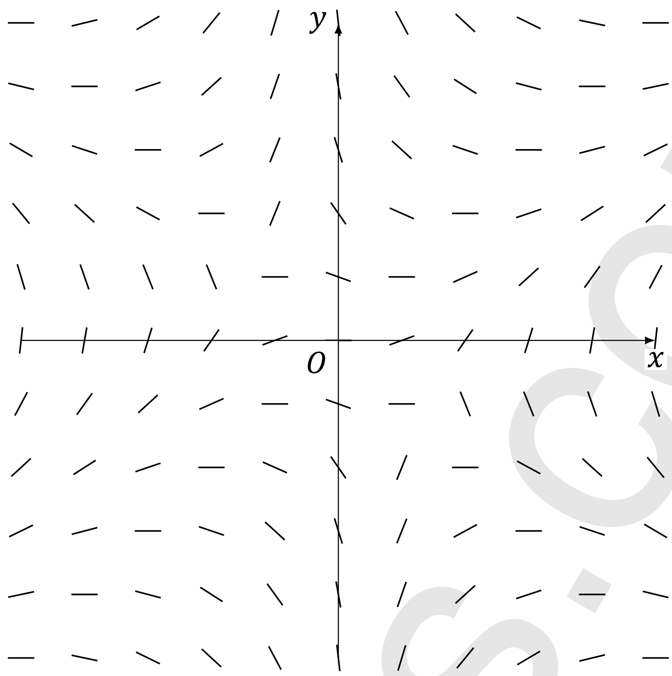
A. $y = 6\sin^{-1} x + \dfrac{5\pi}{4}$

B. $y = 6\sin^{-1} x + \dfrac{\pi}{2}$

C. $y = \dfrac{3}{2}\sin^{-1} x + \dfrac{5\pi}{4}$

D. $y = \dfrac{3}{2}\sin^{-1} x + \dfrac{\pi}{2}$

8 Consider the following slope field:



Which of the following options is a possible equation for this slope field?

A. $\dfrac{dy}{dx} = \dfrac{x^2 - y^2}{2xy - 1}$

B. $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy - 1}$

C. $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy + 1}$

D. $\dfrac{dy}{dx} = \dfrac{x^2 - y^2}{2xy + 1}$

9 Which of the following gives the value of $\sin^{-1}(\cos x)$ if $n\pi < x < (n+1)\pi$, where n is an odd integer?

A. $n\pi - x$

B. $n\pi - x + \dfrac{\pi}{2}$

C. $x - n\pi$

D. $x - n\pi - \dfrac{\pi}{2}$

10 Four distinct integers from 1 to 100 are selected at random. What is the probability they can form an arithmetic progression?

A. $\dfrac{1}{2025}$

B. $\dfrac{1}{2225}$

C. $\dfrac{1}{2425}$

D. $\dfrac{1}{2625}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hours 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use the Question 11 Writing Booklet

(a) Find the integral $\int \frac{8}{\sqrt{9-4x^2}}\,dx$ **2**

(b) Solve the equation $2\sin\theta = 1 + \cos\theta$ for $0 \leq \theta \leq 2\pi$. **3**

(c) Show that $f(x) = \tan^{-1} x - x$, defined for all real x, has an inverse function. **2**

(d) The working life of a particular brand of electric light bulb has a mean of 1200 hours and a standard deviation of 200 hours, and can be assumed to be normally distributed. Zarif is an electrician and purchases 64 light bulbs. What is the probability that at least 3 of these bulbs fail within 1000 hours? **3**

(e) Solve the inequality $\frac{x^2-2}{x^2-3} \leq 2$. **3**

(f) Let the roots of $x^3 - 2\pi x^2 + 5\pi^2 x - \pi^3$ be α, β, and γ. **1**

Find the value of $(\alpha - \pi)(\beta - \pi)(\gamma - \pi)$.

End of Question 11

Question 12 (14 marks) Use the Question 12 Writing Booklet

(a) Adrian runs a ski resort in the chilly province of Harbin, in China. He models the relationship between N, the number of tourists at his resort, and T °C, the air temperature, with the equation **2**

$$N = 2930 - (T + 440)\,[\ln(T + 49)]^2\,.$$

At a certain moment, it is known that the air temperature is -40°C, and is falling at a rate of half a degree per hour. Find the rate of change of the number of tourists at that moment.

(b) The differential equation **2**

$$\frac{dP}{dt} = k(P - 10)$$

where k is a constant models the population $P(t)$ (in millions of bacteria) of a bacterial colony after t hours. Given that $P(1) = 1$ and $P(3) = 5$, find $P(6)$.

(c) Consider the function $f(x) = \tan^{-1} x$.

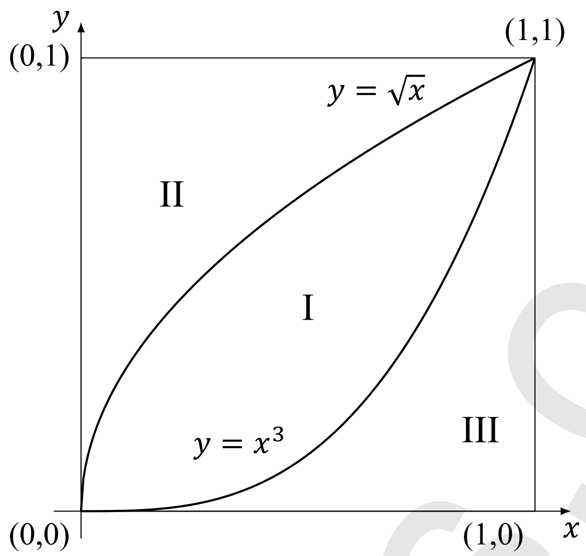
(i) Sketch the function $y = (f(x))^2$, showing all asymptotes and intercepts. **2**

(ii) Hence, sketch the function $y = \dfrac{1}{(f(x))^2}$, showing all asymptotes and intercepts. **2**

Question 12 continues on page 7

Question 12 (continued)

(d) A unit square is divided into three regions by the curves $y = \sqrt{x}$ and $y = x^3$ as shown.



Cathy prints the unit square out and uses it as a peculiarly shaped dartboard. She throws darts randomly, so you may assume every point on the square is equally likely to be hit, and that she will never miss the square.

(i) Find the probability that she hits region I. **1**

(ii) Cathy is unhappy with this probability. She creates a new version of the board, replacing the curve $y = x^3$ with the curve $y = x^r$, where $2r > 1$. Region I remains the region bound by the two curves. **2**

Find the value of r if the probability she hits region I is now $\dfrac{29}{75}$.

(e) The vectors $\begin{pmatrix} a+2b \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2a-b \\ 3 \end{pmatrix}$ are known to be perpendicular, and have the same magnitude. Find all possible values of a. **3**

End of Question 12

Question 13 (14 marks) Use the Question 13 Writing Booklet

(a) Suppose that m is a positive integer.

(i) Give a combinatorial explanation to why the product of m consecutive integers is divisible by $m!$. **2**

(ii) Hence, if n is a positive integer, use mathematical induction to prove that $(3n)!$ is divisible by 6^n. **3**

(b) Two particles A and B are launched from the same point on horizontal ground. Particle A is launched with speed U at an angle α, whilst particle B is launched T seconds later with speed V at an angle β.

The position vectors of the particles, $\underset{\sim}{r}_A(t)$ and $\underset{\sim}{r}_B(t)$, where t is the time after particle A was launched and g is the acceleration due to gravity, are given by

$$\underset{\sim}{r}_A(t) = (Ut\cos\alpha)\underset{\sim}{i} + \left(Ut\sin\alpha - \frac{gt^2}{2}\right)\underset{\sim}{j}, \qquad \text{(Do NOT prove this.)}$$

$$\underset{\sim}{r}_B(t) = (V(t-T)\cos\beta)\underset{\sim}{i} + \left(V(t-T)\sin\beta - \frac{g(t-T)^2}{2}\right)\underset{\sim}{j}.$$

Suppose the particles collide at time $t = t_0$.

(i) Show that $t_0 = \dfrac{VT\cos\beta}{V\cos\beta - U\cos\alpha}$. **1**

(ii) Hence, by finding a second expression for t_0, show that **3**

$$T = \frac{2UV\sin(\alpha - \beta)}{g(U\cos\alpha + V\cos\beta)}.$$

Question 13 continues on page 9

Question 13 (continued)

(c) Ismaeel and Rex arrive at a parking lot with 20 parking spaces, as shown.

1	2	3	$\cdots$	18	19	20

They agree to pick two different spaces at random (each pair of spaces is equally likely to be chosen), and park their Ferrari and Lamborghini there.

(i) How many different ways are there for Ismael and Rex to park such that their cars have at most two spaces in between them? **1**

(ii) Ismaeel and Rex decide to go to a new parking lot, which has 3 rows of 20 parking spaces arranged strangely as shown. **1**

41	42	43	$\cdots$	58	59	60

21	22	23	$\cdots$	38	39	40

1	2	3	$\cdots$	18	19	20

If they decide where to park in the same way, what is the probability that their cars have at most two spaces in between them? Parking in spaces 20 and 60 does NOT count as having one space between the cars.

(iii) Given that Ismaeel and Rex are parked with at most two spaces between them, what is the probability that there is a car in a parking space ending in 0? **3**

End of Question 13

Question 14 (18 marks) Use the Question 14 Writing Booklet

(a) *You may use the information on page 12 to answer this question.* **4**

The NSWMaths discord server is hosting a mock exam. All members who purchase a ticket are guaranteed to come, and each person is allowed to invite a maximum of one friend for free to the exam.

The probability that xeach paying member will bring a friend is 50%. However, as the event venue has a capacity of 60 people, ticket sales must be limited to prevent a scenario where the venue becomes exceeds capacity.

Find the maximum number tickets that can be sold while ensuring that there is less than a 1% probability of exceeding the venue's capacity.

(b) Let J be the coefficient of x^3 in $(x^2 + 8x + 7)^{11}$. Without using a calculator, find the remainder J has when divided by 6. **3**

(c) Suppose that $\underset{\sim}{p}, \underset{\sim}{q}$, and $\underset{\sim}{r}$ are three nonzero vectors.

It is known that $\underset{\sim}{p} + \underset{\sim}{q} + \underset{\sim}{r} = \hat{\underset{\sim}{p}} + \hat{\underset{\sim}{q}} + \hat{\underset{\sim}{r}} = \underset{\sim}{0}$.

(i) If λ is the real number satisfying $\hat{\underset{\sim}{r}} = \lambda(\underset{\sim}{p} + \underset{\sim}{q})$, show that $\lambda = -\dfrac{1}{|\underset{\sim}{p} + \underset{\sim}{q}|}$. **1**

(ii) Show that $\left(\dfrac{|\underset{\sim}{p}| - |\underset{\sim}{p} + \underset{\sim}{q}|}{|\underset{\sim}{p}|}\right)\underset{\sim}{p} = \left(\dfrac{|\underset{\sim}{p} + \underset{\sim}{q}| - |\underset{\sim}{q}|}{|\underset{\sim}{q}|}\right)\underset{\sim}{q}$. **1**

(iii) Hence, prove that $\underset{\sim}{p}, \underset{\sim}{q}$, and $\underset{\sim}{r}$ are vectors representing the edges of an equilateral triangle. **3**

Question 14 continues on page 11

Question 14 (continued)

(d) Hugh has $n \geq 1$ coins in his left and n coins in his right pocket. He will choose a pocket with equal probability and donate a coin from that pocket to NSW Mathematics.

Hugh will continue to do this until one of his pockets is empty, and will keep the rest of the coins. Thus the number of coins left is given by some random variable X.

Let the probability that $X = k$ be given by $P(k)$.

(i) Explain why the probability that Hugh will end with k coins is given by **2**

$$P(k) = \begin{cases} \binom{2n-k-1}{n-k} \dfrac{1}{2^{2n-k-1}}, & \text{if } k = 1, 2, \cdots, n, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Prove the relationship **1**

$$2(n-k)P(k) = (2n-k-1)P(k+1).$$

(iii) By considering $P(1) + P(2) + \cdots + P(n)$, or otherwise, show that **2**

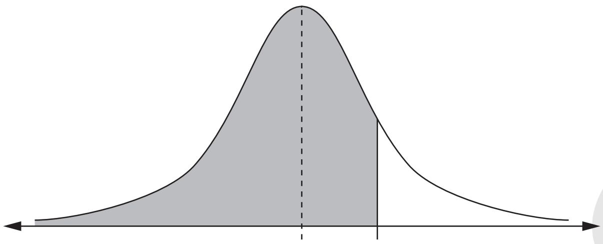
$$E(X) = (2n-1)P(1).$$

(iv) Hence, deduce the inequality $\binom{2n}{n} \leq 2^{2n-1}$. **2**

End of paper

You may use the information below to answer Question 14 (a).

Table of values $P(Z \leq z)$ for the normal distribution $N(0,1)$



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995

i think my story is almost exactly the same as everyone else's, minus a few tweaks.
i am not the exception,
just an extrapolation.

— Angie Wang, private conversation.

EXAMINERS

one eight seven	Q2, 7, 9, 11[a, b, c, e], 12c
amiyuki7	Q10, 13b
Hugh Entwistle	Q14d
MagiicMushroom	Everything else.

NSW GOVERNMENT

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2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \; r \neq 1$$

$$S = \frac{a}{1 - r}, \; |r| < 1$$

Logarithmic and Exponential Functions

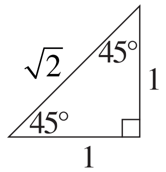
$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

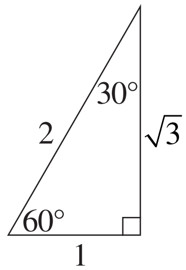
Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$



$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

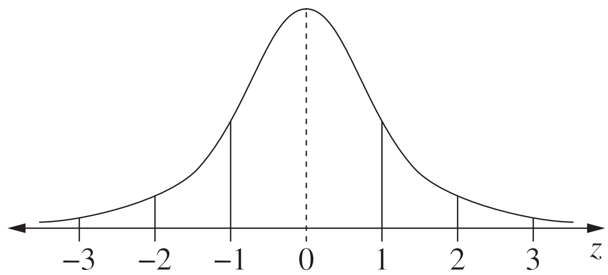
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

Q1

$$\frac{6}{13}\begin{pmatrix}-34\\-6\end{pmatrix}+\frac{7}{13}\begin{pmatrix}18\\7\end{pmatrix}=\begin{pmatrix}-6\\1\end{pmatrix}$$

C

Q2

Coeff $x^2 = -1\times 4 = -4$

Coeff $x = +1\times -5 = -5$

Constant $= -1\times 6 = -6$

A

Q3

Total $= 12!$

Unfavourable $= 9!\times\binom{10}{3}\times 3!$

x _ x _ x _ x _ x _ x _ x _ x _ x _ x

choose positions

$$\Rightarrow \text{Prob} = 1-\frac{9!\times\binom{10}{3}\times 3!}{12!}=\frac{5}{11}$$

arrange moderators

B

Q4

$X\sim B(10,0.4)$

$$P(X<3\mid X>1)=\frac{P(X<3\cap X>1)}{P(X>1)}$$

$$=\frac{P(X=2)}{1-P(X\le 1)}$$

$$=\frac{\binom{10}{2}(0.4)^2(0.6)^8}{1-0.6^{10}-\binom{10}{1}(0.4)(0.6)^9}\approx 0.1268$$

B

Q5

$$\int_x^{4x} \frac{1}{t+\sqrt{t}}\,dt$$

$u = \sqrt{t}$

$2du = \frac{1}{\sqrt{t}}\,dt$

t	x	$4x$
u	$\sqrt{x}$	$2\sqrt{x}$

$$= \int_x^{4x} \frac{1}{\sqrt{t}(\sqrt{t}+1)}\,dt$$

$$= \int_{\sqrt{x}}^{2\sqrt{x}} \frac{2}{u+1}\,du$$

$$= \left[2\ln|u+1|\right]_{\sqrt{x}}^{2\sqrt{x}}$$

$$= 2\ln\left(\frac{2\sqrt{x}+1}{\sqrt{x}+1}\right)$$

A

Q6

Be careful of vector orientation!

$$\cos(180^\circ - \alpha) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}$$

$$\alpha = 180^\circ - \arccos\left(\frac{17871}{187 \cdot 781}\right)$$

$$\simeq 97^\circ$$

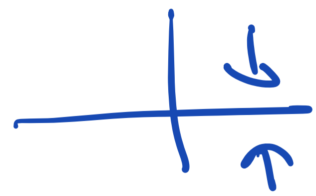
A

Q7

$$y = 3\arccos x + \pi$$

Reflection about y-axis:

$$y = 3\arccos(-x) + \pi$$
$$= 3(\pi - \arccos x) + \pi$$
$$= -3\arccos x + 4\pi$$



Dilate vertically by $\frac{1}{2}$

$$y = -\frac{3}{2}\arccos x + 2\pi$$

Translate down by $\frac{3\pi}{4}$

$$y = -\frac{3}{2}\arccos x + \underbrace{\left(2\pi - \frac{3\pi}{4}\right)}_{\frac{5\pi}{4}}$$

D

Q8

Try: $(2,1)$

A: $\frac{2^2 - 1^2}{4-1} = 1$ More

B: $\cancel{\frac{1^2 - 2^2}{4-1} = -1}$

C: $\cancel{\frac{1^2 - 2^2}{4+1} = -\frac{3}{5}}$

D: $\frac{2^2 - 1}{4+1} = \frac{3}{5}$ Less

D

Q9

$\arcsin(\cos x)$

$= \arcsin\left(\sin\left(\frac{\pi}{2} - x\right)\right)$

$= \arcsin\left(-\sin\left(\frac{\pi}{2} - x + n\pi\right)\right)$

$\sin(2\pi + x) = \sin x$

$\sin(\pi + x) = -\sin x$

For odd n

$\sin(n\pi + x) = -\sin x$

$= -\left(\frac{\pi}{2} - x + n\pi\right)$

$= x - n\pi - \frac{\pi}{2}$

$\arcsin(\sin x) = x$

if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\arcsin\left(\sin\left(\frac{\pi}{2} - x\right)\right) = x$

if $-\frac{\pi}{2} \le \frac{\pi}{2} - x \le \frac{\pi}{2}$

$-\pi \le -x \le 0$

$0 \le x \le \pi$

$n\pi < x < (n+1)\pi$

$0 < x - n\pi < \pi$

$-\pi < n\pi - x < 0$

$\boxed{-\frac{\pi}{2} < n\pi + \frac{\pi}{2} - x < \frac{\pi}{2}}$

D

Q10

Total: $\binom{100}{4}$

Favorable:

$d=1 \longrightarrow$ $(1,2,3,4), (2,3,4,5), \ldots (97,98,99,100)$
$97 = 100 - 3\times 1$

$d=2 \longrightarrow$ $(1,3,5,7), (2,4,6,8), \ldots (94,96,98,100)$
$94 = 100 - 3\times 2$

$d=3 \longrightarrow$ $(1,4,7,10), (2,5,8,11), \ldots, (91,94,97,100)$
$91 = 100 - 3\times 3$

$d=4 \rightarrow (1,5,9,13), \ldots, (88,92,96,100)$

$88 = 100 - 3\times 4$

$\vdots$

$d=33 \rightarrow (1, 34, 67, 100)$

$1 = 100 - 3\times 33$

Add: $33\times 100 - 3(1+2+\cdots+33)$

$$= 3300 - 3\times \frac{33\times 34}{2}$$

$$= 1617$$

Final ans: $\dfrac{1617}{\binom{100}{4}} = \dfrac{1}{2425}$ (C)

Question 11

a) $\displaystyle\int \frac{8}{\sqrt{9-4x^2}}\,dx = \int \frac{8}{2\sqrt{\frac{9}{4}-x^2}}\,dx$

$= 4\arcsin\left(\frac{2x}{3}\right)+C$

b) $2\sin\theta = 1+\cos\theta$

$\cos\theta - 2\sin\theta = -1$

$\cos\theta - 2\sin\theta \equiv R\cos(\theta+\alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$R\cos\alpha = 1 \quad — ①$

$R\sin\alpha = 2 \quad — ②$

$①^2+②^2: \quad R^2(\cos^2\alpha+\sin^2\alpha) = 1^2+2^2$

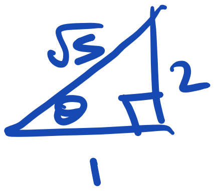
$R = \sqrt{5}$

$② \div ① : \quad \tan\alpha = 2$

$\alpha = \arctan 2$

$0 \le \theta \le 2\pi$

$\cos(\theta+\arctan 2) = \frac{1}{\sqrt{5}}$ $\quad \arctan 2 \le \theta + \arctan 2 \le 2\pi + \ldots$

Note: 

from triangle

$$\arctan 2 = \arccos \frac{1}{\sqrt{5}}$$

don't tell aniyuki7

$$\theta + \arccos\frac{1}{\sqrt{5}} = \pi - \arccos\frac{1}{\sqrt{5}}, \pi + \arccos\frac{1}{\sqrt{5}}$$

$$\Rightarrow \theta = \pi - 2\arccos\frac{1}{\sqrt{5}}, \pi.$$

c) $f(x) = \arctan x - x$

$$f'(x) = \frac{1}{1+x^2} - 1$$

$$= \frac{1 - 1 - x^2}{1+x^2}$$

$$= \frac{x^2}{1+x^2}.$$

Since $x^2 \le 0$ (and $1+x^2 > 0$),

$f'(x) \le 0$, which means

$f(x)$ is monotonic decreasing on its entire domain, and is therefore one-to-one.

Hence $f(x)$ is invertible.

Note $f(x)$ attain only one stationary point (at $x=0$), hence it will still satisfy the horizontal line test.

d) $X \sim N(1200, 200)$

P(1 light bulb fails within 1000 hr)

$= P(X \leq 1000)$

$= P\left(Z \leq \frac{1000-1200}{200}\right)$

$= P(Z \leq -1)$

$= 1 - P(Z \leq 1)$

$= 1 - 0.8413$

$= 0.1587 \longrightarrow p$

$Y \sim B(64, 0.1587)$, want

$P(Y \geq 3) = 1 - 0.8413^{64} - \binom{64}{1} 0.8413^{63} \cdot 0.1587$

$- \binom{64}{2} 0.8413^{62} \cdot 0.1587^2$

≈ 0.999984

e) $\dfrac{x^2-2}{x^2-3} \le 2$ $\longrightarrow$ $x^2-3 \ne 0$

$x \ne \pm\sqrt{3}$

$\cancel{\dfrac{x^2-3}{x^2-3}}^{\,1} + \dfrac{1}{x^2-3} \le 2$

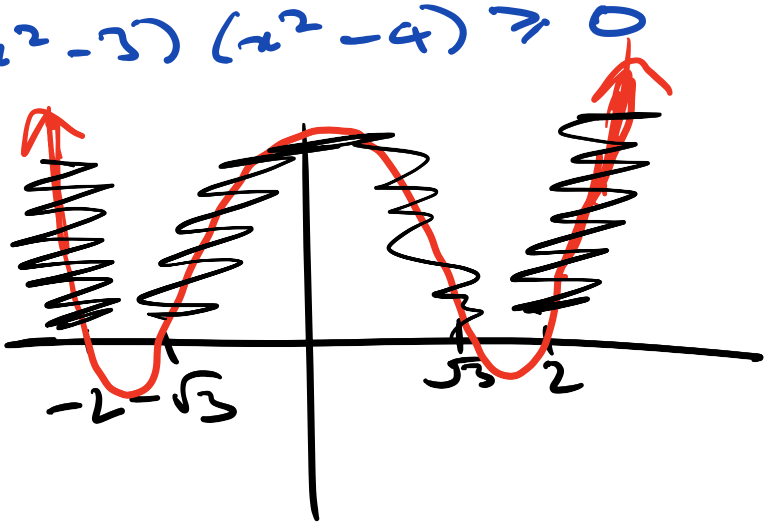
$\dfrac{1}{x^2-3} \le 1$

$x^2-3 \le (x^2-3)^2$

$(x^2-3)^2 - (x^2-3) \geqslant 0$

$(x^2-3)\left[(x^2-3)-1\right] \geqslant 0$

$(x^2-3)(x^2-4) \geqslant 0$



$\therefore$ $x \le -2$, $-\sqrt{3} < x < \sqrt{3}$, $x \geqslant 2$

f) Consider

$$(x+\pi)^3 - 2\pi(x+\pi)^2 + 5\pi^2(x+\pi) - \pi^3$$

$$= x^3 + (\ldots)x^2 + (\ldots)x + \pi^3 - 2\pi(\pi^2) + 5\pi^2(\pi) - \pi^3$$

$$= x^3 + \overset{\text{idc}}{\ldots\ldots} + 3\pi^3$$

Using above polynomial, we see

$$(\alpha-\pi)(\beta-\pi)(\gamma-\pi) = -\frac{3\pi^3}{1}.$$

Question 12

a) $N = 2930 - (T+440)\left[\ln(T+49)\right]^2$

When $T = -40$, $\frac{dT}{dt} = -\frac{1}{2}$

$\Rightarrow$ When $T = 40$,

$$\frac{dN}{dt} = \frac{dN}{dT}\frac{dT}{dt}$$

$$= \frac{1}{2}\left(\left[\ln(T+49)\right]^2 + (440+T)\frac{2\ln(T+49)}{T+49}\right)$$

$$\approx 100.068$$

$\therefore$ Increasing at 100 (approx) tourists per hour.

b) $$\frac{dP}{dt} = k(P-10)$$

$\longrightarrow$ $P = 10$ is a constant solution to the DE, but is not what we require, because $P(1) = 1 \neq 10$.

$$k\,dt = \frac{dP}{P-10}$$

$$\int k\,dt = \int \frac{dP}{P-10}$$

$$kt = \ln|P-10| + c$$

Since $P(1) = 1$, we require **negative** case for abs. value.

$P(1) = 1$ $\quad k = \ln 9 + c$ — ①

$P(3) = 5$. $\quad 3k = \ln 5 + c$ — ②

② − ①: $2k = \ln \frac{9}{5}$

$k = \frac{1}{2} \ln \frac{9}{5}$

sub into ①: $\frac{1}{2} \ln \frac{9}{5} = \ln 9 + c$

$\therefore \frac{t}{2} \ln \frac{9}{5} = \ln|P-10| + \frac{1}{2} \ln \frac{9}{5} - \ln 9$

For P(6):

$3 \ln \frac{9}{5} = \ln|P-10| + \frac{1}{2} \ln \frac{9}{5} - \ln 9$

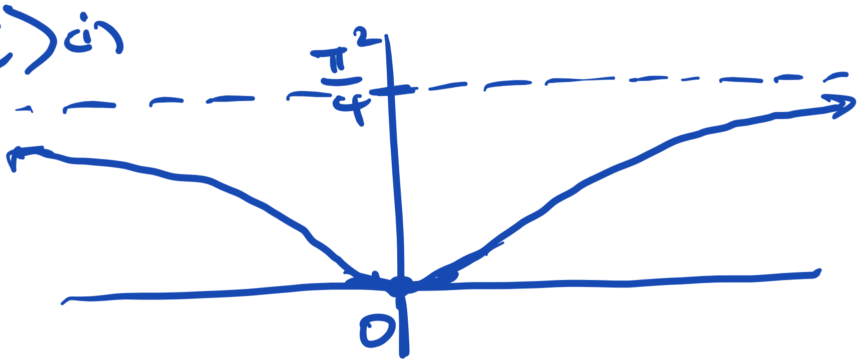
$\frac{5}{2} \ln \frac{9}{5} - \ln 9 = \ln(10-P)$

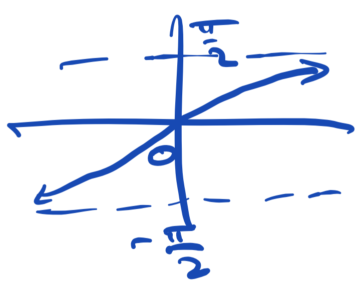
$$\frac{e^{\frac{5}{2}\ln \frac{9}{5}}}{e^{\ln 9}} = 10 - P$$

$$P = 10 - \frac{\left(\frac{9}{5}\right)^{\frac{5}{2}}}{9}$$

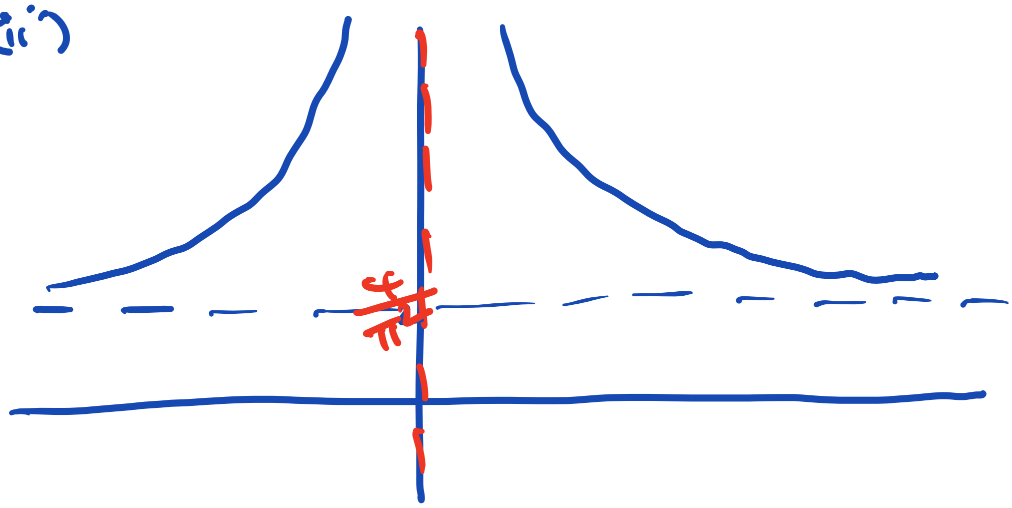
$\simeq 9.517$

(c)(i)





(ii)



d) (i) Area of region I

$$= \int_0^1 \sqrt{x} - x^3 \, dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{5}{12}$$

$$\text{Prob of landing in I} = \frac{\text{Area of I}}{\underbrace{\text{Area of unit square}}_{=1\times 1 = 1}}$$

$$= \frac{5}{12}$$

(ii) Following similar idea,

$$\frac{29}{75} = \int_0^1 \sqrt{x} - x^n \, dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^{n+1}}{n+1} \right]_0^1$$

$$\frac{29}{75} = \frac{2}{3} - \frac{1}{r+1}$$

$$\frac{1}{r+1} = \frac{7}{25}$$

$$r = \frac{18}{7}$$ one eight / seven

e) $$0 = \begin{pmatrix} a+2b \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2a-b \\ 3 \end{pmatrix}$$ (perpendicular)

$$0 = 12 + 2a^2 + \overset{3ab}{\cancel{4ab - ab}} - 2b^2 \quad \text{— ①}$$

$$\left| \begin{pmatrix} a+2b \\ 4 \end{pmatrix} \right|^2 = \left| \begin{pmatrix} 2a-b \\ 3 \end{pmatrix} \right|^2$$

$$(a+2b)^2 + 16 = (2a-b)^2 + 9$$

$$7 = (2a-b)^2 - (a+2b)^2$$
$$= [(2a-b)-(a+2b)][(2a-b)+(a+2b)]$$
$$= (a-3b)(3a+b)$$
$$= 3a^2 - 8ab - 3b^2 \quad \text{— ②}$$

①×3 || $$-36 = 6a^2 + 9ab - 6b^2 \quad \text{— ③}$$

②×2 || $$14 = 6a^2 - 16ab - 6b^2 \quad \text{— ④}$$

③ − ④ $\quad -50 = 25ab$

$ab = -2$ ——— ⑤

sub in ⑤

①: $-12 = 2a^2 + 3ab - 2b^2$

$= 2a^2 - 6 - 2b^2$

$-6 = 2a^2 - 2b^2$

$-3 = a^2 - b^2$

$b = -\frac{2}{a}$

$-3 = a^2 - \frac{4}{a^2}$

$0 = a^4 + 3a^2 - 4$

$= (a^2 - 1)(a^2 + 4)$ → no solutions

$a = \pm 1$

Question 13

a)(i)
m consecutive integers can always be written in the form $n+1, n+2, \ldots, n+m$. for some integer $n \geq 0$

But: $\binom{n+m}{m} = \frac{(n+m)!}{m!\,n!}$ $\qquad n! = n(n-1)! = n(n-1)(n-2)!$

$$= \frac{(n+m)(n+m-1)\cdots(n+2)(n+1)\cancel{n!}}{m!\,\cancel{n!}}$$

$$= \frac{(n+m)(n+m-1)\cdots(n+2)(n+1)}{m!}$$

The numerator is the product of the aforementioned consecutive integers. But we know $\binom{n+m}{m}$ is always an integer.

Therefore, the required product of integers must be divisible by $m!$.

(ii) For $n=1$:

$(3n)! = 3! = 6 = 6^1 \times 1.$ ✓

Assume for $n=k$:

$(3k)! = 6^k M$ for another $M \in \mathbb{Z}$.

Then for $n=k+1$,

$$(3(k+1))!$$
$$= (3k+3)!$$
$$= (3k+3)(3k+2)(3k+1)\ (3k)!$$
$$= (3k+3)(3k+2)(3k+1)\ (6^k M) \quad \text{(assumption)}$$
$$= (6N)(6^k M) \quad \text{for some } N \in \mathbb{Z}$$

because $3k+3$, $3k+2$, $3k+1$ are 3 consecutive integers, hence by part (i) is divisible by $3! = 6$

$$= 6^{k+1}(MN)$$

which is divisible by 6^{k+1}.

Hence the statement holds by mathematical induction.

b) (i)

$$\underset{\sim}{r}_A(t_0) = \underset{\sim}{r}_B(t_0)$$

$$\Rightarrow \quad Ut_0 \cos\alpha = V(t_0 - T)\cos\beta \quad (1)$$

$$Ut_0 \sin\alpha - \frac{gt_0^2}{2} = V(t_0 - T)\sin\beta - \frac{g(t_0 - T)^2}{2} \quad (2)$$

Rearrange (1):

$$Ut_0\cos\alpha = Vt_0\cos\beta - VT\cos\beta$$

$$VT\cos\beta = Vt_0\cos\beta - Ut_0\cos\alpha$$
$$= t_0(V\cos\beta - U\cos\alpha)$$
$$t_0 = \frac{VT\cos\beta}{V\cos\beta - U\cos\alpha}. \quad \leftarrow 3$$

(ii) Rearrange 2:

$$Ut_0\sin\alpha - \cancel{\frac{gt_0^2}{2}} = Vt_0\sin\beta - VT\sin\beta - \cancel{\frac{gt_0^2}{2}} + gt_0T - \frac{gT^2}{2}$$

$$VT\sin\beta + \frac{gT^2}{2} = Vt_0\sin\beta - Ut_0\sin\alpha + gt_0T$$
$$= t_0(V\sin\beta - U\sin\alpha + gT)$$

$$t_0 = \frac{VT\sin\beta + \frac{gT^2}{2}}{V\sin\beta - U\sin\alpha + gT} \quad \leftarrow 4$$

$$\therefore \frac{V\cancel{T}\cos\beta}{V\cos\beta - U\cos\alpha} = \frac{V\cancel{T}\sin\beta + \frac{gT^{\cancel{2}}}{2}}{V\sin\beta - U\sin\alpha + gT}$$

can cancel T as T > 0
(B launched after A)

$$V\cos\beta\,(V\sin\beta - U\sin\alpha + gT)$$

$$= (V\cos\beta - U\cos\alpha)\left(V\sin\beta + \frac{gT}{2}\right)$$

$$\cancel{V^2\cos\beta\sin\beta} - UV\sin\alpha\cos\beta + gVT\cos\beta$$

$$= \cancel{V^2\cos\beta\sin\beta} - UV\cos\alpha\sin\beta$$

$$+ \frac{gVT\cos\beta}{2} - \frac{gUT\cos\alpha}{2}$$

$$\frac{gVT\cos\beta + gUT\cos\alpha}{2} = UV\sin\alpha\cos\beta - UV\cos\alpha\sin\beta$$

!!!

$$gT(V\cos\beta + U\cos\alpha) = 2UV(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

$$T = \frac{2UV\sin(\alpha-\beta)}{U\cos\alpha + V\cos\beta}.$$

c)(i) Cases:

- Right next to each other: 2×19
- Gap 1: 2×18
- Gap 2: 2×17

$\Rightarrow$ 108 ways

(ii) $P(\leq 2 \text{ spaces})$

$= P(\leq 2 \text{ spaces} \mid \text{same row})\, P(\text{same row})$

$\cancel{\underbrace{P(\leq 2 \text{ spaces} \mid \text{different rows})}_{=0}\, P(\text{different rows})}$

use (i) → $= \dfrac{108}{\binom{20}{2} \times 2!} \times \dfrac{19}{59}$ $\binom{20}{2} \times 2! \rightarrow {}^{20}P_2$

assume replacement

$= \dfrac{27}{295}.$

(iii) $P(\text{car in a X0 spot} \mid \leq 2 \text{ spaces apart})$

$= \dfrac{P(\overbrace{\text{car in } 10, 20, 30, 40, 50, \text{ or } 60}^{\text{car in } 10,30,50 \; + \; \text{car in } 20,40,60} \cap \leq 2 \text{ spaces apart})}{P(\leq 2 \text{ spaces apart})}$ ← use (ii)

Favorable outcomes:

- Case 1: car in 20, 40, 60.

If car in 20, then other car must be in 19, 18, or 17.

→ 3×2

choices for car positions   arrange two cars

Cases car in 40 & 60 are identical

→ Total is $3 \times 3 \times 2 = 18$.

- Case 2: car in 10, 30, 50.

This is the same idea, except the other car can now be on the other side. e.g.

7 8 9 10 11 12 13

→ Total is $2 \times 18 = 36$

Hence combined favorable outcomes = 54.

$$\therefore \text{Prob} = \frac{\dfrac{54}{2!\binom{60}{2}}}{\dfrac{27}{295}} = \frac{1}{6}$$

ALTERNATE FASTER APPROACH

Numerator: # favorable outcomes (i.e. 10, 20, ..., or 60 picked) arising 2 spaces apart

$= 54$

Denominator: # outcomes where the cars are 2 spaces apart.

$= 3 \times 108 = 324$ → $\frac{54}{324} = \frac{1}{6}$

Question 14

a) Let $n =$ number of tickets sold.

Attendance $= n + X$,

$X \sim B(n, 0.5)$ (X is the number of people who bring a friend).

We want $P(n + X > 60) < 0.01$

$$P(X > 60 - n) < 0.01$$

$$1 - P(X \leq 60 - n) < 0.01$$

$$P(X \leq 60 - n) > 0.99$$

Approximate X by $N\left(\frac{n}{2}, \frac{n}{4}\right)$, $\quad n = n, \; p = \frac{1}{2}$

then if $Z \sim N(0,1)$, we have

$$P\left(Z \leq \frac{(60-n) - \frac{n}{2}}{\sqrt{\frac{n}{4}}}\right) > 0.99$$

table

$$\frac{(60-n) - \frac{n}{2}}{\sqrt{\frac{n}{4}}} > 2.33$$

$$60 - \frac{3n}{2} > 2.33\sqrt{\frac{n}{4}}$$

$$\frac{3n}{2} + 2.33\sqrt{\frac{n}{4}} - 60 < 0$$

$$1.5n + 1.165\sqrt{n} - 60 < 0$$

For equality,

$$\sqrt{n} = \frac{-1.165 \pm \sqrt{1.165^2 + 360}}{3}$$

reject negative case as $\sqrt{n} > 0$

$$n = (5.94813\ldots)^2$$

$$= 35.38028\ldots$$

$\Rightarrow$ At most 35 people.

b) $(x^2 + 8x + 7)^{11}$

$$= [(x^2 + 2x + 1) + \underline{6}(x+1)]^{11}$$

$$= \binom{11}{0}(x^2+2x+1)^{11} + \binom{11}{1}(x^2+2x+1)^{10}[6(x+1)]$$

$$+ \binom{11}{2}(x^2+2x+1)^{9}[6(x+1)]^2$$

$$+ \ldots$$

We want to consider the coefficient of x^3, but only the remainder when divided by 6.

TRICK: Irrespective of the coefficient of x^3 in $\binom{11}{1}(x^2+2x+1)^{10}[6(x+1)]$,

$\binom{11}{2}(x^2+2x+1)^{9}[6(x+1)]^2$, ...,

its remainder will always be $\underline{\underline{0}}$,

because 6 is a factor of the entire term.

Hence we only need the coeff of x^3 in the first term $(x^2+2x+1)^{11} = (x+1)^{22}$

$$\Rightarrow \binom{22}{3} = \frac{22!}{3!\,19!}$$

$$= \frac{22\times21\times20}{6}$$

$$= 22\times7\times10$$

$$= 1540$$

know: $1500 = \frac{3000}{2} = 6\times250$

$\therefore$ The remainder is the same as the remainder for $40 \div 6$, which is 4.

c) $\underset{\sim}{p} + \underset{\sim}{q} + \underset{\sim}{r} = \underset{\sim}{0}$

$\hat{\underset{\sim}{p}} + \hat{\underset{\sim}{q}} + \hat{\underset{\sim}{r}} = \underset{\sim}{0}$

(i) $\underset{\sim}{r} = -(\underset{\sim}{p} + \underset{\sim}{q})$

$$\hat{\underset{\sim}{r}} = \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|}$$

$$= \frac{-(\underset{\sim}{p} + \underset{\sim}{q})}{|-(\underset{\sim}{p} + \underset{\sim}{q})|}$$

$$= -\frac{1}{|\underset{\sim}{p} + \underset{\sim}{q}|}(\underset{\sim}{p} + \underset{\sim}{q})$$

$$\Rightarrow \lambda = -\frac{1}{|\underset{\sim}{p} + \underset{\sim}{q}|}$$

(ii) $\underset{\sim}{0} = \hat{\underset{\sim}{p}} + \hat{\underset{\sim}{q}} + \hat{\underset{\sim}{r}}$

$$= \hat{\underset{\sim}{p}} + \hat{\underset{\sim}{q}} - \frac{\underset{\sim}{p} + \underset{\sim}{q}}{|\underset{\sim}{p} + \underset{\sim}{q}|}$$

$$= \frac{\underset{\sim}{p}}{|\underset{\sim}{p}|} + \frac{\underset{\sim}{q}}{|\underset{\sim}{q}|} - \frac{\underset{\sim}{p}}{|\underset{\sim}{p} + \underset{\sim}{q}|} - \frac{\underset{\sim}{q}}{|\underset{\sim}{p} + \underset{\sim}{q}|}$$

$$\underset{\sim}{p}\left(\frac{1}{|\underset{\sim}{p} + \underset{\sim}{q}|} - \frac{1}{|\underset{\sim}{p}|}\right) = \underset{\sim}{q}\left(\frac{1}{|\underset{\sim}{q}|} - \frac{1}{|\underset{\sim}{p} + \underset{\sim}{q}|}\right)$$

$$\underset{\sim}{p}\left(1 - \frac{|\underset{\sim}{p} + \underset{\sim}{q}|}{|\underset{\sim}{p}|}\right) = \underset{\sim}{q}\left(\frac{|\underset{\sim}{p} + \underset{\sim}{q}|}{|\underset{\sim}{q}|} - 1\right)$$

$$\underline{p}\left(\frac{|\underline{p}|-|\underline{p}+\underline{q}|}{|\underline{p}|}\right)=\underline{q}\left(\frac{|\underline{p}+\underline{q}|-|\underline{q}|}{|\underline{q}|}\right).$$

(iii) Case 1 $\hat{\underline{p}}, \hat{\underline{q}}$ parallel:

Suppose $\hat{\underline{p}}, \hat{\underline{q}}$ parallel, then they are scalar multiple of one another, BUT they are both unit vectors.

So:

Case $\hat{\underline{p}}=\hat{\underline{q}}$	$\hat{\underline{p}}=-\hat{\underline{q}}$
$\hat{\underline{r}}=-\hat{\underline{p}}-\hat{\underline{p}}$	$\hat{\underline{r}}=-\hat{\underline{p}}+\hat{\underline{p}}$
$=-2\hat{\underline{p}}$	$=\underline{0}$

Both cases are contradiction because unit vectors can only have magnitude 1, and this suggests magnitude 2 or 0.

Case 2: $\hat{\underline{p}}, \hat{\underline{q}}$ non-parallel

From (ii),

$$(|\underline{p}|-|\underline{p}+\underline{q}|)\,\hat{\underline{p}}=(|\underline{p}+\underline{q}|-|\underline{q}|)\,\hat{\underline{q}}.$$

Because $\hat{p}$ and $\hat{q}$ are not parallel, they are not linearly independent, hence can't be non-zero scalar multiples of each other.

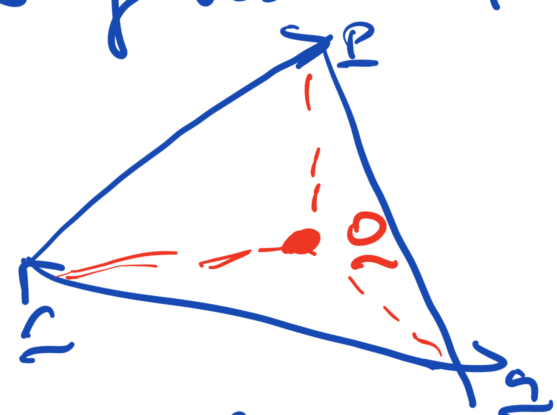
$$\therefore \quad |p| - |p+q| = 0$$

$$\Rightarrow |p| = |p+q| = |r|$$

AND $|p+q| - |q| = 0$

$$\Rightarrow |q| = |p+q| = |r|$$

$\Rightarrow$ hence all sides are equal, but also the sum of vectors $p + q + r = 0$.



$\Rightarrow$ p, q, r form sides of an equilateral triangle.

d) (i)

$P(k) = 0$ otherwie is obvios becase can't have negtive # of coins left and can't have more coins than what you started with.

(Also cannot finish on 0 coins.)

For $k = 1, 2, \dots, n$, first note it does not matter which pocket has remaing coins.

(Because: if he ends up with k coins in his left pocket, by simply inverting which pocket he draws from, he will have the same amount of coins in the right pocket.)

$\therefore$ Assume WLOG coins end up in left pocket

$P(k$ in left pocket$) = P\Big((n-1)$ coins taken out of right pocket AND $n-k$ coins taken out of left pocket IN ANY ORDER$\Big) \times P($1 out of right pocket$)$

$$= \binom{(n-1)+(n-k)}{n-k} \left(\frac{1}{2}\right)^{n-k} \left(\frac{1}{2}\right)^{n-1} \times \frac{1}{2}$$

$$= \binom{2n-k-1}{n-k} \frac{1}{2^{2n-k-1}} \times \frac{1}{2}$$

P(k is right pocket) = P(k is left pocket)

$$\Rightarrow P(k) = \binom{2n-k-1}{n-k} \frac{1}{2^{2n-k-1}}$$

(ii)

$$2(n-k)P(k) = (n-k)\binom{2n-k-1}{n-k}\frac{1}{2^{2n-k-2}}$$

$$= \cancel{(n-k)} \frac{(2n-k-1)!}{(n-k)\cancel{!}\,(n-1)!} \frac{1}{2^{2n-k-2}}$$

$$= \frac{(2n-k-1)!}{(n-k-1)!\,(n-1)!} \frac{1}{2^{2n-k-2}}$$

$$= (2n-k-1) \frac{(\overbrace{2n-k-2}^{=2n-(k+1)-1})!}{(\underbrace{n-k-1}_{=n-(k+1)})!\,(n-1)!} \frac{1}{2^{2n-k-2}}$$

$$= (2n-k-1)\binom{2n-(k+1)-1}{n-(k+1)}\frac{1}{2^{2n-(k+1)-1}}$$

$$\sum_k c\,a_k = c\sum_k a_k \qquad = (2n-k-1)\,P(k+1) \qquad \sum_k (a_k+b_k) = \sum_k a_k + \sum_k b_k$$

(iii)

$$\sum_{k=1}^{n} 2(n-k)P(k) = \sum_{k=1}^{n} (2n-k-1)P(k+1)$$

$$2n\sum_{k=1}^{n} P(k) - 2\sum_{k=1}^{n} kP(k) = 2n\sum_{k=1}^{n} P(k+1) - \sum_{k=1}^{n} (k+1)P(k+1)$$

BUT: $\sum_{k=1}^{n} P(k) = 1$

AND! $\sum_{k=1}^{n} k P(k) = E(X)$ by definition.

$\therefore$ LHS $= 2n - 2E(X)$

Then: $\sum_{k=1}^{n} P(k+1)$

$$= P(2) + P(3) + \cdots + P(n) + \underbrace{P(n+1)}_{=0}$$

$$= -P(1) + \left[P(1) + P(2) + P(3) + \cdots + P(n)\right]$$

$$= 1 - P(1)$$

Similarly: $\sum_{k=1}^{n} (k+1) P(k+1)$

$$= 2P(2) + 3P(3) + \cdots + nP(n) + \underbrace{(n+1)P(n+1)}_{=0}$$

$$= -1P(1) + \left[1P(1) + 2P(2) + 3P(3) + \cdots + nP(n)\right]$$

$$= E(X) - P(1)$$

$$\Rightarrow \text{RHS} = 2n\left(1 - P(1)\right) - \left(E(X) - P(1)\right)$$

$$= 2n - 2n P(1) - E(X) + P(1)$$

Putting everything together,

$$\cancel{2n} - 2E(x) = \cancel{2n} - 2n\,P(1) - E(x) + P(1)$$

$$-E(x) = (-2n+1)\,P(1)$$

$$E(x) = (2n-1)\,P(1).$$

(iv) $$P(1) = \binom{2n-2}{n-1}\frac{1}{2^{2n-2}}$$

$$\Rightarrow E(x) = (2n-1)\,\frac{(2n-2)!}{[(n-1)!]^2}\,\frac{1}{2^{2n-2}}$$

$$= \frac{(2n-1)!}{[(n-1)!]^2}\,\frac{1}{2^{2n-2}}$$

$$= \frac{(2n)!}{(n!)^2}\,\frac{n^2}{2n}\,\frac{1}{2^{2n-2}}$$

$$= \binom{2n}{n}\frac{n}{2^{2n-1}}$$

We know that $E(x) \le n$ always holds, because n is the max value of this random variable.

$$\therefore \binom{2n}{n}\frac{\cancel{n}}{2^{2n-1}} \le \cancel{n} \Rightarrow \binom{2n}{n} \le 2^{2n-1}.$$