2025 **MOCK HSC EXAM**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 5–11)

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Suppose $z = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$, and $w = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$. The value of $\operatorname{Arg}\left(\dfrac{z^2}{w^2}\right)$ is:

A. $\dfrac{7\pi}{6}$

B. $\dfrac{5\pi}{6}$

C. $-\dfrac{\pi}{6}$

D. $-\dfrac{5\pi}{6}$

2 Suppose that f is a continuous, positive function, with domain all real x.

Define $g(x) = \displaystyle\int_{e^{-x}}^{e^{x}} f(t)\,dt$. Which of the following is always true about g?

A. There exists a real number c such that $g(x) = c$ has no roots.

B. For all real c, there exists exactly one x with $g(x) = c$.

C. g has an inverse on its whole domain, and $g^{-1}(0)$ always exists.

D. For all real $c, g(x) = c$ has multiple solutions.

3 Franz whispers two propositions from the shadowed corridors of his mind into your ear:

A: “If you obey, you endure. If you do not obey, you will not endure.”
B: “If you do not submit, you cannot obey.”

Suppose **A** is known to be true. Which of the following must then be logically equivalent to **B**?

A. If I endure, then I will submit.

B. If I submit, then I will not obey.

C. If I endure, then I will not obey.

D. If I submit, then I will endure.

4 A particle is moving in simple harmonic motion between two points, L and U, with centre O. Given that the particle's speed when it reaches O is $\sqrt{2}\ \mathrm{ms^{-1}}$, what is the particle's speed when it is halfway between O and L?

A. $\dfrac{\sqrt{3}}{\sqrt{2}}\ \mathrm{m\,s^{-1}}$ B. $\dfrac{\sqrt{15}}{2}\ \mathrm{m\,s^{-1}}$ C. $\dfrac{5\sqrt{3}}{3}\ \mathrm{m\,s^{-1}}$ D. $\dfrac{4\sqrt{3}}{\sqrt{2}}\ \mathrm{m\,s^{-1}}$

5 Consider the point $P(2,2,3)$, and the line interval ℓ with equation $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

for $|\lambda| \leq 1$. What is the shortest distance from ℓ to P?

A. 3 B. $\sqrt{3}$ C. $2\sqrt{2}$ D. $3\sqrt{2}$

6 It is known that a polynomial $P(z)$ of degree 13 with real coefficients has a factorisation

$$P(z) = (z^5 + 4)(z - 1 - i)(z^2 - 2z + 2)(z - 1)Q(z),$$

for some polynomial $Q(z)$ with complex coefficients.

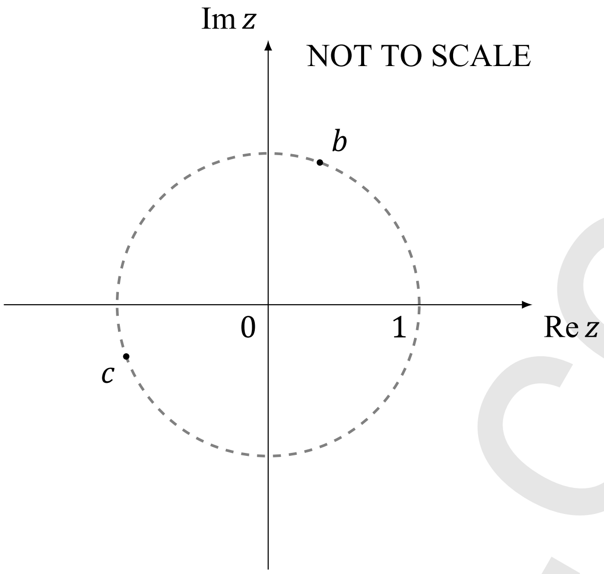
What is the minimum number of real roots (counting multiplicity) $P(z)$ could have?

A. 2 B. 3 C. 4 D. 6

7 The value of $\displaystyle\int_0^{\pi} \frac{x\cos(3x)}{\cos(x)}\, dx$ is:

A. $\dfrac{\pi^2}{2}$ B. $\dfrac{\pi^2 + 1}{2}$ C. $\dfrac{1 - \pi^2}{2}$ D. $-\dfrac{\pi^2}{2}$

8 Suppose that b and c are complex numbers on the following circle.



It is known that c is a root to the equation $z^2 + az + b = 0$, where a is a complex number. Which of the following is the maximum possible value of $|a|$?

A. 1 B. $1 - \sqrt{2}$ C. $\sqrt{2}$ D. 2

9 There are five tiles on the board, showing

A, H, 6, 7, BLUE

Each tile has a letter on one side and either a number or a colour on the other **(but not both)**.

Suppose that we need to check that following rule is satisfied:

"If a tile shows a vowel, then the other side is an even number or a colour."

What is the minimum number of tiles needed to check, in order to conclude that the rules have been satisfied?

A. 1 B. 2 C. 3 D. 4

10 Define, for $x \in [1, 87]$, the function $f(x) = \int_1^x \frac{(\ln(t))^2 - 2\ln(x) + 3}{t}\, dt$.

Which of the following is closest to the maximum of $f(x)$?

A. $\frac{14}{3}$ B. $\frac{4}{3}$ C. $\frac{4}{5}$ D. $\frac{16}{5}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Let $z = 4 + i$ and $w = 2 - i$.

(i) Express $\dfrac{\overline{z}}{w} - \dfrac{\overline{w}}{z}$, in the form $x + iy$, where x and y are real numbers. **2**

(ii) Express $(z - w)^{12}$ in the form $x + iy$, where x and y are real numbers. **2**

(b) Let $f(x) = \dfrac{1}{(x+1)(2x+3)}$.

(i) Find constants A and B such that $f(x) = \dfrac{A}{x+1} + \dfrac{B}{2x+3}$. **1**

(ii) Hence, show the volume of the surface of revolution generated by rotating the region under $f(x)$ from $x = 0$ to $x = 1$ is equal to $4\pi \ln \dfrac{5}{6} + \dfrac{23\pi}{30}$ u^3. **3**

(c) Sketch the region defined by $\mathrm{Re}\left(\dfrac{z - i}{z + i}\right) = 1$ on the Argand diagram. **3**

(d) A particle is moving in a straight line such that its displacement from a fixed point, O, at time t is given by $x = 3 - 4\cos(3t) + 3\sin(3t)$.

(i) Show that the particle is moving in simple harmonic motion. **1**

(ii) What is the maximum velocity of the particle? **1**

(iii) When is the displacement of the particle zero for the first time? **2**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Let a, b, and c be nonnegative real numbers where $a + b + c = 1$. Prove that **1**

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \leq 1.$$

(b) Consider the statement S about a differentiable function f:

S: "If $\int_{-1}^{1} (1+x)f'(x)\,dx = 0$, then $f(x)$ is not an odd function or $f(1) = 0$."

(i) State the contrapositive of S. **1**

(ii) Hence, prove that S is true. **3**

(iii) Determine whether the converse of S is true. **1**

(c) A vector $\underset{\sim}{v}$ in 3D satisfies $|\underset{\sim}{v}|\underset{\sim}{i} = \underset{\sim}{v} + 4\underset{\sim}{a}$, where $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$.

(i) Find $|\underset{\sim}{v}|$. **2**

(ii) Find the angle between $\underset{\sim}{v}$ and $\underset{\sim}{a}$. **2**

(d) Let x be a six digit number where $a, b, c, d, e, f \in \{1, 2, 3, 4, 5, 6\}$ correspond to the digits of x (so written out, x would read $abcdef$). Note that the digits $0, 7, 8$ and 9 are **not** used.

(i) Suppose x is divisible by 11. Show that the number $a - b + c - d + e - f$ is also divisible by 11. **2**

(ii) Hence or otherwise prove that at least two of the digits of x must be the same. **3**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Show that none of the roots of $z^6 + z^3 + 1$ lie on the locus of $|z - 1| = |z + 2|$. **2**

(b) Let C be the locus of all possible positions of a point P on the Cartesian plane, given that the distance from P to the point $(-1, 1)$ is equal to the perpendicular distance of P from the line $y = x$.

(i) Let the position vector of P be $\begin{pmatrix} x_p \\ y_p \end{pmatrix}$. Show, using vector methods, that **3**

$$(x_p + 1)^2 + (y_p - 1)^2 = \frac{(x_p - y_p)^2}{2}$$

is an equation for C.

(ii) Show that C is tangent to coordinate axis. **2**

(iii) By sketching C, or otherwise, find the area C encloses with the coordinate axis. **4**

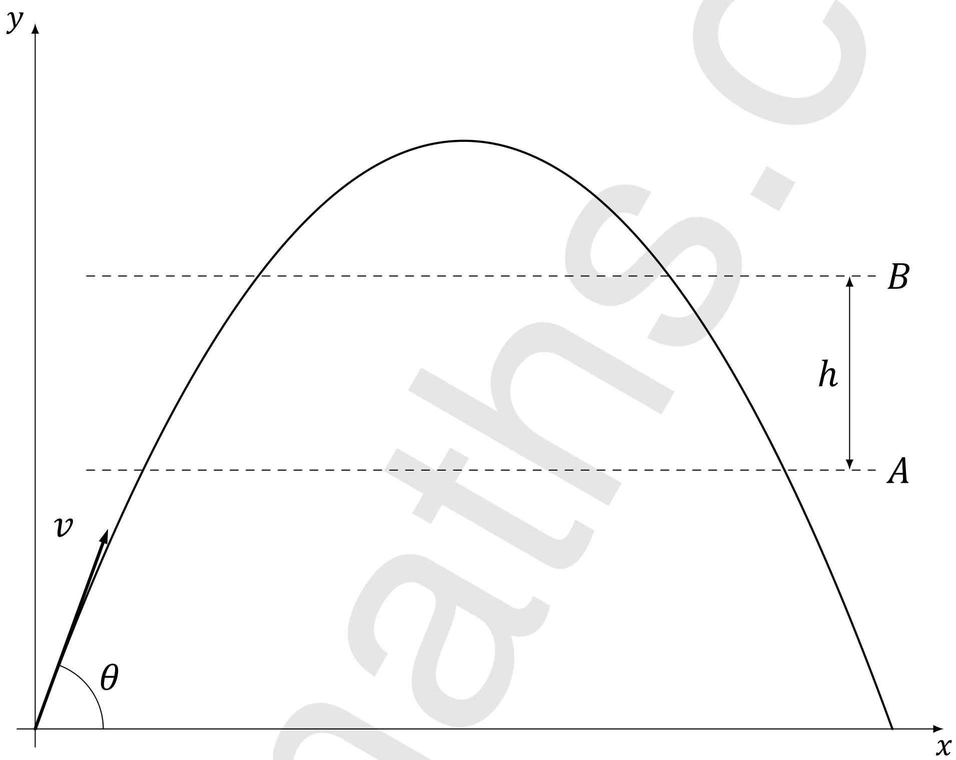
Question 13 continues on page 8

Question 13 (continued)

(c) A projectile P is projected at an angle θ to the positive x-axis, at an initial velocity v. You may assume that the equation of motion for the projectile is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} vt\cos\theta \\ vt\sin\theta - \dfrac{gt^2}{2} \end{pmatrix}, \qquad \text{(Do NOT prove this.)}$$

where g is the acceleration of gravity.



Lines $y = A$ and $y = B$ have a vertical separation of h. The time between P s two crossings of lines A and B are T_A and T_B respectively.

(i) If $\alpha > \beta$ are the real two roots of $bx - ax^2 = c$, find an expression for $\alpha - \beta$. **1**

(ii) Show that $g = \dfrac{8h}{T_A^2 - T_B^2}$. **3**

End of Question 13

Question 14 (16 marks) Use the Question 14 Writing Booklet

(a) (i) Solve the inequality $\dfrac{\sqrt{x+1}}{\sqrt{x}-\sqrt{x-1}} > x$ for x. **1**

(ii) Prove by mathematical induction that if n is a positive integer, then **3**

$$\frac{\sqrt{2}}{1}+\frac{\sqrt{3}}{2}+\frac{\sqrt{4}}{3}+\cdots+\frac{\sqrt{n+1}}{n} > \sqrt{n}.$$

(iii) By considering the graph of $y = \dfrac{\sqrt{x}}{x}$ for $x \geq 1$, prove without mathematical induction that if n is a positive integer, then **3**

$$\frac{\sqrt{2}}{1}+\frac{\sqrt{3}}{2}+\frac{\sqrt{4}}{3}+\cdots+\frac{\sqrt{n+1}}{n} > 2\sqrt{n+1}-2.$$

(b) An object is initially at rest A units away from a point P. It moves towards P due to gravitational acceleration that is inversely proportional to the square of the distance x of the object from P. Define positive to be away from P.

When the particle is x units away from P, it travels with speed v units towards P.

(i) Prove that **2**

$$v = -\sqrt{\frac{2k(A-x)}{Ax}}$$

where k is some constant of proportionality.

(ii) Let T be the number of seconds it takes for the object to reach the half-way point between its initial position and P. **4**

Show that $T = \dfrac{A\sqrt{A}(\pi+2)}{4\sqrt{2k}}$.

(c) Using the substitution $u = x\sqrt{x}$, evaluate $\displaystyle\int_0^1 \sqrt{\frac{3x}{3+x^3}}\,dx$. **3**

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) Suppose $n \in \mathbb{Z}^-$. Let $I_n = \int_0^{\frac{\pi}{2}} (\sin x + 1)^n \, dx$.

(i) Show that **3**

$$I_n = I_{n+1} - 1 - n \int_0^{\frac{\pi}{2}} \cos^2 x (1 + \sin x)^{n-1} \, dx.$$

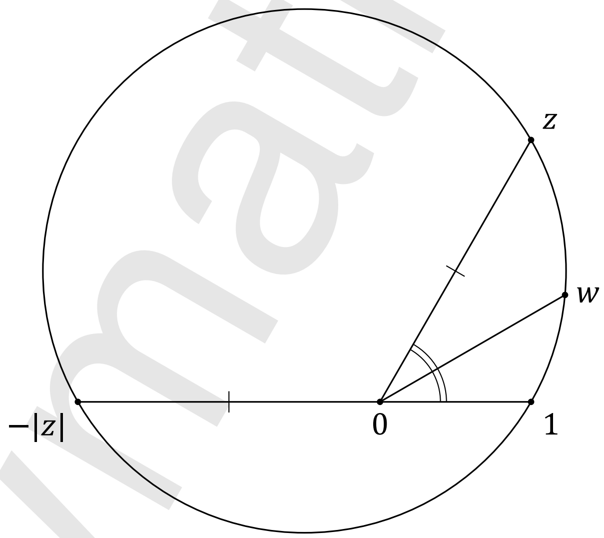
(ii) Hence, show that **3**

$$I_n = \frac{n+1}{2n+1} I_{n+1} - \frac{1}{2n+1}.$$

(iii) Show that $\int_0^{\frac{\pi}{2}} \frac{dx}{(1 + \sin x)^4} = \frac{12}{35}$. **1**

(b) Let z and w be two complex numbers in the first quadrant of the Argand diagram, with $\operatorname{Arg}(z) = 2\operatorname{Arg}(w)$. It is known that there exists a circle which passes through the four complex numbers $1, z, w$, and $-|z|$.

This information is shown in the following diagram.



(i) Explain why there is a positive real number k satisfying $kw^2 = z$. **1**

(ii) By considering two equal angles subtended by the same arc, prove that **3**

$$\frac{(z-1)(w+|z|)}{(w-1)(z+|z|)}$$

is purely real.

(iii) Deduce that w is one of the square roots of z. **4**

Question 16 (14 marks) Use the Question 16 Writing Booklet

(a) Suppose $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$ are nonzero 3D vectors satisfying the following three conditions:

1. $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a} - \underset{\sim}{b}| = |\underset{\sim}{c}|$,
2. $|\underset{\sim}{a}| = 3|\underset{\sim}{b}|$, and
3. $(2\underset{\sim}{a} + \underset{\sim}{b})$ is parallel to $\underset{\sim}{c}$.

(i) Show that $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$. **1**

(ii) Explain why there exists a real number λ satisfying $2\underset{\sim}{a} + \underset{\sim}{b} = \lambda\underset{\sim}{c}$, and show further that λ satisfies **2**

$$\lambda^2 = \frac{4|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2}{|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2}.$$

(iii) Find all possible values of $\underset{\sim}{c}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **2**

(b) Suppose $n \geq 2$ is a positive integer, and let $\omega = e^{\frac{\pi i}{n}}$ be an $2n$th root of unity. Consider the polynomial $P(z) = z^{2n} - 1$.

(i) Show that $2n = \left|(1-\omega)(1-\omega^2)\cdots(1-\omega^{2n-1})\right|$. **2**

(ii) Hence, show that $\displaystyle\sum_{k=1}^{n-1} \ln\left(\sin\frac{\pi k}{2n}\right) = \frac{1}{2}\ln n - (n-1)\ln 2$. **4**

(c) Consider the function $f(x) = \ln(\csc x)$ for $x \in \left(0, \frac{\pi}{2}\right]$. **3**

You may assume that $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} = 0$ (Do NOT prove this).

Find an expression for S_n, the sum of the areas of n lower rectangles for $f(x)$ over the interval $\left(0, \frac{\pi}{2}\right]$, and hence conclude using (b) that

$$\int_0^{\frac{\pi}{2}} \ln(\csc x)\, dx = \frac{\pi}{2}\ln 2.$$

End of paper

To look back one day, through all the sorrow,
the celebration, the desperation, the 3AM pleas on ears that were tired of listening,
See the paths I didn't take,
But appreciate the one I did choose.
To tell people I lost: it was quite fortunate to have met you.
To tell myself that the clear blue night would come.
To find forgiveness for carrying the hopes and dreams of the dead,
for they would want you to carry your own for the future.

Look forward to the once obscured starlight,
the shadows it casts dimming with every step towards sunrise.

EXAMINERS

neutde	Q2, 4, 5
one eight seven	Q11[a,d], Q12[b,c], Q13a
Hugh Entwistle	Q9, Q12d, Q14b
MagiicMushroom	Everything else.

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}\left[2a + (n - 1)d\right] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

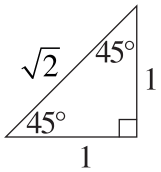
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

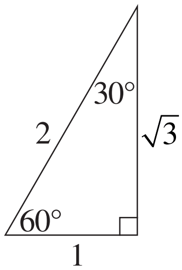
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

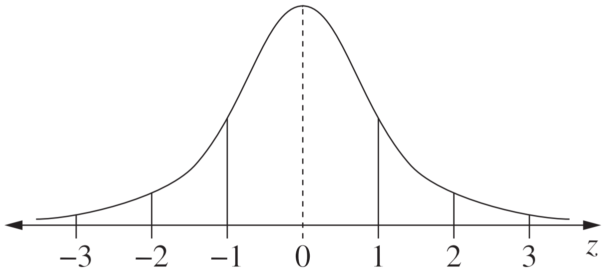
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x) = \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

Solutions

by MagiicMushroom

2025 MOCK HSC EXAM

Mathematics Extension 2

Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–8)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 9–34)

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Suppose $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, and $w = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. The value of $\text{Arg}\left(\frac{z^2}{w^2}\right)$ is:

A. $\frac{7\pi}{6}$ B. $\frac{5\pi}{6}$ C. $-\frac{\pi}{6}$ **D.** $-\frac{5\pi}{6}$

Notice, $z = e^{\frac{i\pi}{4}}$ and $w = e^{-\frac{i\pi}{3}}$, so $\frac{z^2}{w^2} = e^{\frac{i\pi}{2}+\frac{2i\pi}{3}} = e^{\frac{7i\pi}{6}}$. Since Arg is restricted between $-\pi$ and π, we must subtract one multiple of 2π and thus have

$$\text{Arg}\left(\frac{z^2}{w^2}\right) = \frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}. \boxed{\text{D}}$$

2 Suppose that f is a continuous, positive function, with domain all real x.

Define $g(x) = \int_{e^{-x}}^{e^x} f(t)\,dt$. Which of the following is always true about g?

A. There exists a real number c such that $g(x) = c$ has no roots.

B. For all real c, there exists exactly one x with $g(x) = c$.

C. g has an inverse on its whole domain, and $g^{-1}(0)$ always exists.

D. For all real c, $g(x) = c$ has multiple solutions.

As x increases, $g(x)$ represents integrating a positive function on a larger domain, hence g is an strictly increasing function. Since g is also continuous, it must be one to one from its domain to its range. So D is wrong, and g has an inverse.

Now, depending on the choice of f, g may have range all real numbers (e.g. $f(x) = 1$), or it could be bounded (e.g. $f(x) = (1 + x^2)^{-1}$). Hence, A and B are not always correct.

Finally, $g(0) = \int_1^1 f(x)\,dx = 0$ and thus $g^{-1}(0) = 0$, so $\boxed{\text{C}}$ is correct.

3 Franz whispers two propositions from the shadowed corridors of his mind into your ear:

A: "If you obey, you endure. If you do not obey, you will not endure."
B: "If you do not submit, you cannot obey."

Suppose **A** is known to be true. Which of the following must then be logically equivalent to **B**?

A. If I endure, then I will submit.

B. If I submit, then I will not obey.

C. If I endure, then I will not obey.

D. If I submit, then I will endure.

Let O be "You obey", E be "You endure", S be "You submit". Then, the propositions we are given are:

$A : \{O \implies E, \sim O \implies \sim E\}$, and
$B : \{\sim S \implies \sim O\}$.

Since A is true, then $O \implies E$, and furthermore if we take the contrapositive of $\sim O \implies \sim E$ we find $E \implies O$, so it must be that $O \iff E$, that is O and E are logically equivalent.

Thus, B is logically equivalent to $\sim S \implies \sim E$, and the contrapositive of this is $E \implies S$, which is option $\boxed{\text{A}}$.

4 A particle is moving in simple harmonic motion between two points, L and U, with centre O. Given that the particle's speed when it reaches O is $\sqrt{2}$ ms^{-1}, what is the particle's speed when it is halfway between O and L?

A. $\dfrac{\sqrt{3}}{\sqrt{2}}\,\mathrm{m\,s^{-1}}$ B. $\dfrac{\sqrt{15}}{2}\,\mathrm{m\,s^{-1}}$ C. $\dfrac{5\sqrt{3}}{3}\,\mathrm{m\,s^{-1}}$ D. $\dfrac{4\sqrt{3}}{\sqrt{2}}\,\mathrm{m\,s^{-1}}$

We use $v^2 = -n^2\left((x-c)^2 - A^2\right)$, where A is the amplitude of motion. We know that $c = 0$, and when $x = 0$, we are given that $v = \pm\sqrt{2}$. Substituting, $2 = n^2A^2$. Now, when the particle is halfway between O and A, $x = \frac{A}{2}$, so

$$v^2 = -n^2\left(\frac{A^2}{4} - A^2\right) = \frac{3n^2A^2}{4} = \frac{3}{2},$$

so the speed is equal to $|v| = \sqrt{\frac{3}{2}}\,\mathrm{m\,s^{-1}}$. This is $\boxed{\text{A}}$.

5 Consider the point $P(2,2,3)$, and the line interval ℓ with equation $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for $|\lambda| \leq 1$. What is the shortest distance from ℓ to P?

A. 3 B. $\sqrt{3}$ C. $2\sqrt{2}$ D. $3\sqrt{2}$

The shortest distance is the minimum possible value of $\left| \underset{\sim}{r} - \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right| = \left| \begin{pmatrix} -2+\lambda \\ 2 \\ -2 \end{pmatrix} \right|$. But,

$$\sqrt{(-2+\lambda)^2 + 2^2 + (-2)^2} = \sqrt{(\lambda-2)^2 + 8},$$

and since $\lambda \in [-1, 1]$, the minimum of $(\lambda - 2)^2$ is clearly 1, when $\lambda = 1$. This minimises the whole expression, and one can find the distance at $\lambda = 1$ to be 3, which is $\boxed{\text{A}}$.

6 It is known that a polynomial $P(z)$ of degree 13 with real coefficients has a factorisation

$$P(z) = (z^5 + 4)(z - 1 - i)(z^2 - 2z + 2)(z - 1)Q(z),$$

for some polynomial $Q(z)$ with complex coefficients.

What is the minimum number of real roots (counting multiplicity) $P(z)$ could have?

A. 2 **B.** 3 C. 4 D. 6

$P(z)$ has real coefficients so all its non real roots must come in conjugate pairs. There thus has to be an even amount of them, and since there are 13 roots (counting multiplicity), there must be an odd amount of purely real roots. The only possible option is $\boxed{\text{B}}$.

7 The value of $\displaystyle\int_0^{\pi} \frac{x\cos(3x)}{\cos(x)}\,dx$ is:

A. $\dfrac{\pi^2}{2}$ B. $\dfrac{\pi^2+1}{2}$ C. $\dfrac{1-\pi^2}{2}$ **D.** $-\dfrac{\pi^2}{2}$

The most basic method is to derive

$$\begin{aligned}\cos(3x) &= \cos(2x+x)\\ &= \cos(2x)\cos(x) - \sin(2x)\sin(x)\\ &= (2\cos^2(x)-1)\cos(x) - 2\sin^2(x)\cos(x)\\ &= 2\cos^3(x) - \cos(x) - 2(1-\cos^2(x))\cos(x)\\ &= 4\cos^3(x) - 3\cos(x).\end{aligned}$$

Then,

$$\begin{aligned}\int_0^{\pi} \frac{x\cos(3x)}{\cos(x)}\,dx &= \int_0^{\pi} x(4\cos^2(x)-3)\,dx\\ &= \int_0^{\pi} x\left(4\cdot\frac{1+\cos(2x)}{2}-3\right)\,dx\\ &= 2\int_0^{\pi} x\cos(2x)\,dx - \int_0^{\pi} x\,dx\\ &= \Big[x\sin(2x)\Big]_0^{\pi} - \int_0^{\pi}\sin(2x)\,dx - \frac{\pi^2}{2}\\ &= 0 + \left[\frac{\cos(2x)}{2}\right]_0^{\pi} - \frac{\pi^2}{2}\\ &= -\frac{\pi^2}{2}.\ \boxed{\text{D}}\end{aligned}$$

A more visionary method is to first use kings rule to find:

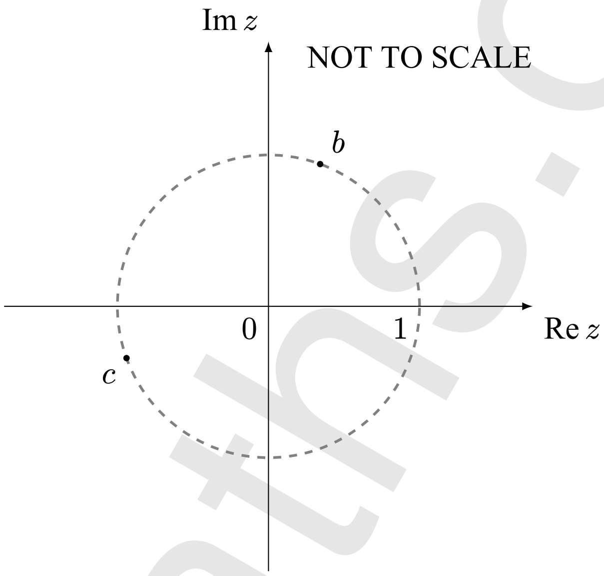
$$\int_0^{\pi}\frac{x\cos(3x)}{\cos(x)}\,dx = \int_0^{\pi}\frac{(\pi-x)\cos(3\pi-3x)}{\cos(\pi-x)}\,dx = \int_0^{\pi}\frac{(\pi-x)\cos(3x)}{\cos(x)}\,dx,$$

so

$$\begin{aligned}2\int_0^{\pi}\frac{x\cos(3x)}{\cos(3x)}\,dx &= \pi\int_0^{\pi}\frac{\cos(3x)}{\cos(x)}\,dx,\\ \int_0^{\pi}\frac{x\cos(3x)}{\cos(x)}\,dx &= \frac{\pi}{2}\int_0^{\pi}\left(\frac{\cos(3x)+\cos(x)}{\cos(x)}-1\right)\,dx\\ &= \frac{\pi}{2}\int_0^{\pi}\frac{\frac{1}{2}\cos(2x)\cos(x)}{\cos(x)} - \frac{\pi}{2}\int_0^{\pi}dx\end{aligned}$$

$$= \frac{\pi}{4}\int_0^{\pi} \cos(2x)\,dx - \frac{\pi^2}{2}$$
$$= \frac{\pi}{8}\Big[\sin(2x)\Big]_0^{\pi} - \frac{\pi^2}{2}$$
$$= -\frac{\pi^2}{2}. \boxed{\text{D}}$$

8 Suppose that b and c are complex numbers on the following circle.



It is known that c is a root to the equation $z^2 + az + b = 0$, where a is a complex number. Which of the following is the maximum possible value of $|a|$?

A. 1 B. $1-\sqrt{2}$ C. $\sqrt{2}$ **D.** 2

Let $z = d$ be the other root of $z^2 + az + b$. From product of roots, $cd = b \implies |cd| = |b|$.

From the diagram, $|b| = |c| = 1$, so we must have $|d| = 1$ too. Then, from sum of roots, $c + d = -a \implies |a| = |c+d| \le |c| + |d| = 2.$ $\boxed{\text{D}}$

(Note this is achievable, e.g. $a = -2i, b = -1, c = d = i$)

9 There are five tiles on the board, showing

$$\text{A}, \text{H}, 6, 7, \text{BLUE}$$

Each tile has a letter on one side and either a number or a colour on the other **(but not both)**.

Suppose that we need to check that following rule is satisfied:

"If a tile shows a vowel, then the other side is an even number or a colour."

What is the minimum number of tiles needed to check, in order to conclude that the rules have been satisfied?

A. 1 **B.** 2 C. 3 D. 4

To show that the statement is true, we need to verify that the A tile has an even number or a colour on the other side. We also need to verify that the tiles without a letter face up which do not have an even number or a colour on them do not have a vowel on their other side (this is us checking that the contrapositive of the statement also holds). So we have to flip over the 7 tile.

We thus flip two cards in total, so $\boxed{\text{B}}$.

10 Define, for $x \in [1, 87]$, the function $f(x) = \displaystyle\int_1^x \frac{(\ln(t))^2 - 2\ln(x) + 3}{t}\, dt$.

Which of the following is closest to the maximum of $f(x)$?

A. $\dfrac{14}{3}$ B. $\dfrac{4}{3}$ C. $\dfrac{4}{5}$ **D.** $\dfrac{16}{5}$

$$\begin{aligned}\int_1^x \frac{(\ln(t))^2 - 2\ln(x) + 3}{t}\, dt &= \int_1^x [(\ln(t))^2 - 2\ln(x) + 3]\frac{d(\ln(t))}{dt}\, dt \\ &= \left[\frac{(\ln(t))^3}{3} - 2\ln(x)\ln(t) + 3\ln(t)\right]_0^x, \\ \therefore f(x) &= \frac{(\ln(x))^3}{3} - 2(\ln(x))^2 + 3\ln(x).\end{aligned}$$

Compute $f'(x)$:

$$\begin{aligned}f'(x) &= \frac{(\ln(x))^2 - 4\ln(x) + 3}{x} \\ &= \frac{(\ln(x) - 1)(\ln(x) - 3)}{x},\end{aligned}$$

So f has stationary points at $x = e$ and $x = e^3$.

A naive student would have substituted these values into our expression for $f(x)$ and found $f(e) = \frac{4}{3}$, and $f(e^3) = 0$. If you proceeded with the second derivative test, you could have even found that $(e, \frac{4}{3})$ is a maximum and $(e^3, 0)$ is a minimum! So $\frac{4}{3}$ must be the maximum of the function and the answer is B.

Except if you study $f'(x) = \dfrac{(\ln(x) - 1)(\ln(x) - 3)}{x}$ more closely, you will notice that f is increasing on $[1, e)$, decreasing on (e, e^3), and increasing on $(e^3, 87]$, from the same way you solve quadratic inequalities.

Indeed, $(e^3, 0)$ is a minimum, so $f(x)$ must be increasing for all x past it! An astute student would have recalled that the global maximum for a continuous, differentiable function occurs either at a local maximum or at an endpoint of the function. We thus have to check our endpoint, $x = 87$. Using your favourite calc (short for calculator), you find $f(87) = 3.1989 \ldots \approx 3.2 = \frac{16}{5}$. And this is the true global maximum of f. $\boxed{\text{D}}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Let $z = 4 + i$ and $w = 2 - i$.

(i) Express $\dfrac{\overline{z}}{w} - \dfrac{\overline{w}}{z}$, in the form $x + iy$, where x and y are real numbers. **2**

Note that if you have a calculator with a complex mode, you can just use it.

$$\begin{aligned}\frac{\overline{z}}{w} - \frac{\overline{w}}{z} &= \frac{4-i}{2-i} - \frac{2+i}{4+i}\\ &= \frac{(4-i)(2+i)}{5} - \frac{(2+i)(4-i)}{17}\\ &= (9+2i)\left(\frac{1}{5} - \frac{1}{17}\right)\\ &= \frac{108}{85} + \frac{24}{85}i.\end{aligned}$$

(ii) Express $(z - w)^{12}$ in the form $x + iy$, where x and y are real numbers. **2**

Note that if you have a calculator with a complex mode, you can just use it.

$$\begin{aligned}(z-w)^{12} &= (4+i-2+i)^{12}\\ &= (2+2i)^{12} = 2^{12}[(1+i)^2]^6\\ &= 2^{12}(2i)^6 = -2^{18}.\end{aligned}$$

(b) Let $f(x) = \dfrac{1}{(x+1)(2x+3)}$.

(i) Find constants A and B such that $f(x) = \dfrac{A}{x+1} + \dfrac{B}{2x+3}$. **1**

$$\begin{aligned}\frac{1}{(x+1)(2x+3)} &= \frac{2x+3-2(x+1)}{(x+1)(2x+3)}\\ &= \frac{1}{x+1} - \frac{2}{2x+3}.\end{aligned}$$

So $A = 1, B = -2$.

(ii) Hence, show the volume of the surface of revolution generated by rotating the region under $f(x)$ from $x = 0$ to $x = 1$ is equal to $4\pi \ln \frac{5}{6} + \frac{23\pi}{30}$ u^3. **3**

Set up the integral to compute the volume.

$$
\begin{aligned}
V &= \pi \int_0^1 f(x)^2 \, dx \\
&= \pi \int_0^1 \left(\frac{1}{x+1} - \frac{2}{2x+3} \right)^2 dx \\
&= \pi \int_0^1 \left(\frac{1}{(x+1)^2} - \underbrace{\frac{4}{(x-1)(2x+3)}}_{4f(x)} + \frac{4}{(2x+3)^2} \right) dx \\
&= \pi \int_0^1 \left(\frac{1}{(x+1)^2} - 4 \left[\frac{1}{x+1} - \frac{2}{2x+3} \right] + \frac{4}{(2x+3)^2} \right) dx \\
&= \pi \left[-\frac{1}{x+1} - 4 \ln|x+1| + 4 \ln|2x+3| - \frac{2}{2x+3} \right]_0^1 \\
&= \pi \left[\left(-\frac{1}{2} - 4\ln 2 + 4 \ln 5 - \frac{2}{5} \right) - \left(-1 - 0 + 4 \ln 3 - \frac{2}{3} \right) \right] \\
&= 4\pi \ln \frac{5}{6} + \frac{23\pi}{30} \text{ u}^3.
\end{aligned}
$$

(c) Sketch the region defined by $\text{Re}\left(\frac{z-i}{z+i} \right) = 1$ on the Argand diagram. **3**

Method 1: Let $z = x + iy$. Notice

$$
\begin{aligned}
\frac{z-i}{z+i} &= \frac{x + i(y-1)}{x + i(y+1)} \\
&= \frac{(x + i(y-1))(x - i(y+1))}{x^2 + (y+1)^2} \\
&= \frac{x^2 + (y-1)(y+1) + i(x(y-1) - x(y+1))}{x^2 + (y+1)^2}.
\end{aligned}
$$

Since we want $\text{Re}\left(\frac{z-i}{z+i} \right) = 1$, it must be that $\frac{x^2 + (y-1)(y+1)}{x^2 + (y+1)^2} = 1$. That is,

$$
\begin{aligned}
x^2 + (y-1)(y+1) &= x^2 + (y+1)^2, \\
y^2 - 1 &= y^2 + 2y + 1, \\
y &= -1.
\end{aligned}
$$

Also note that $z \neq -i$ as then we have division by 0. So the region is just the line $y = -1$ with a hole at $(0, -1)$.

Method 2: Recall that $2\,\mathrm{Re}(w) = w + \bar{w}$. Let $w = \dfrac{z-i}{z+i}$, then

$$\begin{aligned}
\frac{z-i}{z+i} + \frac{\bar{z}+i}{\bar{z}-i} &= 2,\\
\frac{z-i-(z+i)}{z+i} + \frac{\bar{z}+i-(\bar{z}-i)}{\bar{z}-i} &= 0,\\
\frac{2i}{\bar{z}-i} &= \frac{2i}{z+i},\\
\bar{z}-i &= z+i,\\
z-\bar{z} &= -2i.
\end{aligned}$$

Since $z - \bar{z} = 2i\,\mathrm{Im}(z)$, we have $\mathrm{Im}(z) = -1$. Again note the exclusion of $z = -i$.

Method 3 (By M. C. in the exam): Note that $\dfrac{z-i}{z+i} = 1 - \dfrac{2i}{z+i}$, so

$$\mathrm{Re}\left(1 - \frac{2i}{z+i}\right) = 1 \implies \mathrm{Re}\left(\frac{2i}{z+i}\right) = 0.$$

This means that $\dfrac{2i}{z+i}$ is a purely imaginary number, which is only possible if $z+i$ is a nonzero real number. So $\mathrm{Im}(z+i) = 0 \implies \mathrm{Im}(z) = -1$, again $z \neq -i$.

(d) A particle is moving in a straight line such that its displacement from a fixed point, O, at time t is given by $x = 3 - 4\cos(3t) + 3\sin(3t)$.

(i) Show that the particle is moving in simple harmonic motion. **1**

Notice $\dot{x} = 12\sin(3t) + 9\cos(3t)$, and $\ddot{x} = 36\sin(3t) - 27\cos(3t)$. So:

$$\begin{aligned}
\ddot{x} &= -9(3\sin(3t) - 4\cos(3t))\\
&= -3^2(3 - 4\cos(3t) + 3\sin(3t) - 3)\\
&= -3^2(x-3),
\end{aligned}$$

which is in the form of $\ddot{x} = -n^2(x-c)$.

(ii) What is the maximum velocity of the particle? **1**

Let $\dot{x} = 12\sin(3t) + 9\cos(3t) = R\cos(3t - \alpha)$, where $R \geq 0$. Expanding,

$$\begin{aligned}
12\sin(3t) + 9\cos(3t) &= R\cos(3t)\cos(\alpha) + R\sin(3t)\sin(\alpha),\\
\therefore &\begin{cases} 9 = R\cos(\alpha),\\ 12 = R\sin(\alpha),\end{cases}\\
9^2 + 12^2 &= R^2(\cos^2(\alpha) + \sin^2(\alpha)),\\
R &= +15.
\end{aligned}$$

Hence, $\dot{x} = 15\cos(3t - \alpha) \leq 15$, so the maximum velocity is 15 units per time.

(iii) When is the displacement of the particle zero for the first time? **2**

We are solving for $x = 0$ with $t \geq 0$. I.e. we want $4\cos(3t) - 3\sin(3t) = 3$.
Let $4\cos(3t) - 3\sin(3t) = R\cos(3t + \alpha)$, where $R \geq 0$ and $|\alpha| \leq \frac{\pi}{2}$. Expanding,

$$\begin{aligned}
4\cos(3t) - 3\sin(3t) &= R\cos(3t)\cos(\alpha) - R\sin(3t)\sin(\alpha),\\
\therefore &\begin{cases} 4 = R\cos(\alpha),\\ 3 = R\sin(\alpha),\end{cases}\\
4^2 + 3^2 &= R^2(\cos^2(\alpha) + \sin^2(\alpha)),\\
R &= +5,\\
\frac{3}{4} &= \frac{R\sin(\alpha)}{R\cos(\alpha)},\\
\alpha &= \tan^{-1}\frac{3}{4}.
\end{aligned}$$

I realise now it would have been better to do this and then differentiate it for (ii), but oh well. Hence, we have

$$\begin{aligned}
5\cos\left(3t + \tan^{-1}\frac{3}{4}\right) &= 3,\\
3t + \tan^{-1}\frac{3}{4} &= 2k\pi \pm \cos^{-1}\frac{3}{5},\\
t &= \frac{1}{3}\left[2k\pi \pm \cos^{-1}\frac{3}{5} - \tan^{-1}\frac{3}{4}\right],
\end{aligned}$$

and you can verify with a few quick calculator checks that the smallest nonnegative solution for t is $t = \frac{1}{3}\left(\cos^{-1}\frac{3}{5} - \tan^{-1}\frac{3}{4}\right)$. Numerically, this is ≈ 0.0946, if you used t formula or squared both sides.

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Let a, b, and c be nonnegative real numbers where $a + b + c = 1$. Prove that **1**

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \leq 1.$$

Notice that $\dfrac{a}{b^2+1} \leq a$, and similarly for the other two, so

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \leq a + b + c = 1.$$

(b) Consider the statement S about a differentiable function f:

S: "If $\displaystyle\int_{-1}^{1} (1+x)f'(x)\,dx = 0$, then $f(x)$ is not an odd function or $f(1) = 0$."

(i) State the contrapositive of S. **1**

"If $f(x)$ is an odd function and $f(1) \neq 0$, then $\displaystyle\int_{-1}^{1} (1+x)f'(x)\,dx \neq 0$."

(ii) Hence, prove that S is true. **3**

We prove the contrapositive of S is true instead. Assume $f(1) \neq 0$ and f is odd. Approach the integral with integration by parts.

$$\begin{cases} u = 1 + x, & du = dx, \\ dv = f'(x)\,dx, & v = f(x), \end{cases}$$

$$\begin{aligned} \int_{-1}^{1} (1+x)f'(x)\,dx &= \Big[(1+x)f(x)\Big]_{-1}^{1} - \int_{-1}^{1} f(x)\,dx \\ &= 2f(1) - \int_{-1}^{1} f(x)\,dx. \end{aligned}$$

Since f is an odd function, $\displaystyle\int_{-1}^{1} f(x)\,dx$ is the integral of an odd function over a symmetric domain and is equal to zero. Hence, $\displaystyle\int_{-1}^{1} (1+x)f'(x)\,dx = 2f(1) \neq 0$, 得證.

(iii) Determine whether the converse of S is true. **1**

The converse of S is "If $f(x)$ is not an odd function or $f(1) = 0$, then $\displaystyle\int_{-1}^{1} (1+x)f'(x)\,dx = 0$." We prove this is false by counterexample.

Pick $f(x) = x - 1$, then f is not odd **and** $f(1) = 0$, but

$$\int_{-1}^{1} (1+x) f'(x)\, dx = \int_{-1}^{1} (1+x)\, dx = 2 \neq 0,$$

so the implication does not hold.

(c) A vector $\utilde{v}$ in 3D satisfies $|\utilde{v}|\utilde{i} = \utilde{v} + 4\utilde{a}$, where $\utilde{a} = \utilde{i} + 2\utilde{j} - 2\utilde{k}$.

(i) Find $|\utilde{v}|$. **2**

Method 1: Rearrange for $\utilde{v}$ and take the modulus of both sides, so $|\utilde{v}| = ||\utilde{v}|\utilde{i} - 4\utilde{a}|$.

$$\begin{aligned}
|\utilde{v}|^2 &= ||\utilde{v}|\utilde{i} - 54\utilde{a}|^2 \\
&= |\utilde{v}|^2|\utilde{i}|^2 - 8|\utilde{v}|\utilde{i} \cdot \utilde{a} + 16|\utilde{a}|^2, \\
0 &= -8|\utilde{v}| + 16(1^2 + 2^2 + 2^2), \\
|\utilde{v}| &= 18.
\end{aligned}$$

Method 2: Rewrite as a vector equation, so let $\utilde{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and $|\utilde{v}| = \sqrt{x^2+y^2+z^2}$.

Then,

$$\begin{pmatrix} \sqrt{x^2+y^2+z^2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 4\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix},$$

equating components we find that

$$\begin{cases} \sqrt{x^2+y^2+z^2} &= x + 4, \\ 0 &= y + 8, \\ 0 &= z - 8, \end{cases}$$

so $y = -8$, $z = 8$ which we can substitute into the first equation.

$$\begin{aligned}
\sqrt{x^2 + 8^2 + 8^2} &= x + 4, \\
x^2 + 128 &= x^2 + 8x + 16, \\
8x &= 128 - 16, \\
x &= 14.
\end{aligned}$$

Hence, $|\utilde{v}| = \sqrt{14^2 + 8^2 + 8^2} = 18$.

(ii) Find the angle between $\underset{\sim}{v}$ and $\underset{\sim}{a}$. **2**

Method 1: The easiest method, though it requires you to have done (i). Substitute $|\underset{\sim}{v}| = 18$, so $18\underset{\sim}{i} = \underset{\sim}{v} + 4\underset{\sim}{a} \implies \underset{\sim}{v} = 18\underset{\sim}{i} - 4\underset{\sim}{a} = 14\underset{\sim}{i} - 8\underset{\sim}{j} + 8\underset{\sim}{k}$, if you did *Method 2* above you will have arrived at this already, thus the angle between them is given by

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\underset{\sim}{v}\cdot\underset{\sim}{a}}{|\underset{\sim}{v}||\underset{\sim}{a}|}\right)\\ &= \cos^{-1}\left(\frac{14-16-16}{18\sqrt{1+2^2+2^2}}\right)\\ &\approx 109.47^\circ.\end{aligned}$$

Method 2: First, take the magnitude of the equation and square it.

$$\begin{aligned}||\underset{\sim}{v}|\underset{\sim}{i}|^2 &= |\underset{\sim}{v}+4\underset{\sim}{a}|^2,\\ |\underset{\sim}{v}|^2 &= |\underset{\sim}{v}|^2 + 8\underset{\sim}{v}\cdot\underset{\sim}{a} + 16|\underset{\sim}{a}|^2,\\ \underset{\sim}{v}\cdot\underset{\sim}{a} &= -2|\underset{\sim}{a}|^2.\end{aligned}$$

Then, dot product the whole equation with $\underset{\sim}{a}$.

$$\begin{aligned}|\underset{\sim}{v}|\underset{\sim}{i} &= \underset{\sim}{v} + 4\underset{\sim}{a},\\ |\underset{\sim}{v}|\underset{\sim}{i}\cdot\underset{\sim}{a} &= \underset{\sim}{v}\cdot\underset{\sim}{a} + 4|\underset{\sim}{a}|^2,\\ |\underset{\sim}{v}| &= -2|\underset{\sim}{a}|^2 + 4|\underset{\sim}{a}|^2\\ &= 2|\underset{\sim}{a}|^2,\end{aligned}$$

so $\theta = \cos^{-1}\left(\frac{\underset{\sim}{v}\cdot\underset{\sim}{a}}{|\underset{\sim}{v}||\underset{\sim}{a}|}\right) = \cos^{-1}\left(\frac{-2|\underset{\sim}{a}|^2}{2|\underset{\sim}{a}|^3}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{1^2+2^2+2^2}}\right) \approx 109.47^\circ.$

(d) Let x be a six digit number where $a, b, c, d, e, f \in \{1,2,3,4,5,6\}$ correspond to the digits of x (so written out, x would read $abcdef$). Note that the digits $0, 7, 8$ and 9 are **not** used.

(i) Suppose x is divisible by 11. Show that the number $a - b + c - d + e - f$ is also divisible by 11. **2**

Suppose that x is divisible by 11, that is there exists some $N \in \mathbb{Z}$ satisfying $x = 100000a + 10000b + 1000c + 100d + 10e + f = 11N$.

We want to show that $a - b + c - d + e - f$ is also divisible by 11.

$$\begin{aligned}100000a + 10000b + 1000c + 100d + 10e + f &= 11N,\\ 100001a + 9999b + 1001c + 99d + 11e - 11N &= a - b + c - d + e - f,\\ 11(9091a + 909b + 91c + 9d + e - N) &= a - b + c - d + e - f,\end{aligned}$$

so it is indeed divisible by 11.

(ii) Hence or otherwise prove that at least two of the digits of x must be the same. **3**

Assume that a, b, c, d, e, f are all unique. Notice that the maximum of $a-b+c-d+e-f$ is $6-1+5-2+4-3 = 9$, and the minimum is -9. Thus, since $a-b+c-d+e-f$ is divisible by 11, it must equal zero. That is, $a+c+e = b+d+f$. We prove that there is no way to allocate the digits $a, b, \cdots, f$ to $1, 2, \cdots, 6$ in this equation.

Note that $a+b+c+d+e+f = 1+2+3+4+5+6 = 21$ (though maybe not in that order). Using $a+c+e = b+d+f$, we have

$$2(a+c+e) = 21,$$

but the LHS is even (since $a+c+e$ is an integer) and the RHS is odd. This is a contradiction, so we cannot have all distinct digits.

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Show that none of the roots of $z^6 + z^3 + 1$ satisfy $|z-1| = |z+2|$. **2**

We know that the locus $|z-1| = |z+2|$ is the line $\text{Re}(z) = -\frac{1}{2}$. The roots of $z^6 + z^3 + 1 = 0$ are the six roots of $(z^3-1)(z^6+z^3+1) = z^9 - 1 = 0$ where $z^3 \neq 1$, i.e. the six ninth roots of unity that aren't cube roots of unity. That is, $z = e^{\frac{2ik\pi}{9}}$, where $k \in \{1, 2, 4, 5, 7, 8\}$. Since $\text{Re}(z) = \cos\frac{2k\pi}{9}$, we can manually check these six roots to find that none of them have real part $-\frac{1}{2}$, so none lie on the locus.

Note that the restriction of $z^3 \neq 1$ removes the two non real cube roots of unity with real part $-\frac{1}{2}$, specifically $e^{\frac{2i\pi}{3}}$ and $e^{\frac{4i\pi}{3}}$.

(b) Let C be the locus of all possible positions of a point P on the Cartesian plane, given that the distance from P to the point $(-1, 1)$ is equal to the perpendicular distance of P from the line $y = x$.

(i) Let the position vector of P be $\begin{pmatrix} x_p \\ y_p \end{pmatrix}$. Show, using vector methods, that **3**

$$(x_p+1)^2 + (y_p-1)^2 = \frac{(x_p-y_p)^2}{2}$$

is an equation for C.

The distance from $P$to $(-1,1)$ can be expressed as $\left\| \begin{pmatrix} x_p \\ y_p \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} x_p+1 \\ y_p-1 \end{pmatrix} \right\|$. The perpendicular distance from P to the line $y = x$ is given by the perpendicular component of $\begin{pmatrix} x_p \\ y_p \end{pmatrix}$ to a vector parallel to $y = x$, e.g. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This is equal to

$$\begin{aligned}
\left\| \text{perp}_{\binom{1}{1}} \begin{pmatrix} x_p \\ y_p \end{pmatrix} \right\| &= \left\| \begin{pmatrix} x_p \\ y_p \end{pmatrix} - \text{proj}_{\binom{1}{1}} \begin{pmatrix} x_p \\ y_p \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} x_p \\ y_p \end{pmatrix} - \frac{x_p+y_p}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| \\
&= \left\| \frac{1}{2} \begin{pmatrix} 2x_p \\ 2y_p \end{pmatrix} - \frac{1}{2} \begin{pmatrix} x_p+y_p \\ x_p+y_p \end{pmatrix} \right\| \\
&= \frac{1}{2} \left\| \begin{pmatrix} x_p-y_p \\ y_p-x_p \end{pmatrix} \right\|.
\end{aligned}$$

Since we are given that these are the same distance, $\left\| \begin{pmatrix} x_p+1 \\ y_p-1 \end{pmatrix} \right\| = \frac{1}{2} \left\| \begin{pmatrix} x_p-y_p \\ y_p-x_p \end{pmatrix} \right\|$.

Squaring,

$$\begin{aligned}
\left\|\begin{pmatrix} x_p + 1 \\ y_p - 1 \end{pmatrix}\right\|^2 &= \frac{1}{4}\left\|\begin{pmatrix} x_p - y_p \\ y_p - x_p \end{pmatrix}\right\|^2, \\
(x_p + 1)^2 + (y_p - 1)^2 &= \frac{1}{4}\left((x_p - y_p)^2 + (y_p - x_p)^2\right) \\
&= \frac{(x_p - y_p)^2}{2}.
\end{aligned}$$

(ii) Show that C is tangent to coordinate axis. **2**

To show that C is tangent to the x–axis, we solve for the x intercepts of the equation found in part 1, and show that the only intersection is a double root and hence tangential. Setting $y_p = 0$,

$$(x_p + 1)^2 + 1 = \frac{x_p^2}{2} \implies \frac{1}{2}(x_p + 2)^2 = 0,$$

clearly this is a double root so C is tangent to the x–axis. You may repeat this process to convince yourself it is also tangent to the y–axis, or observe the equation in (i) is symmetric about $y = -x$.

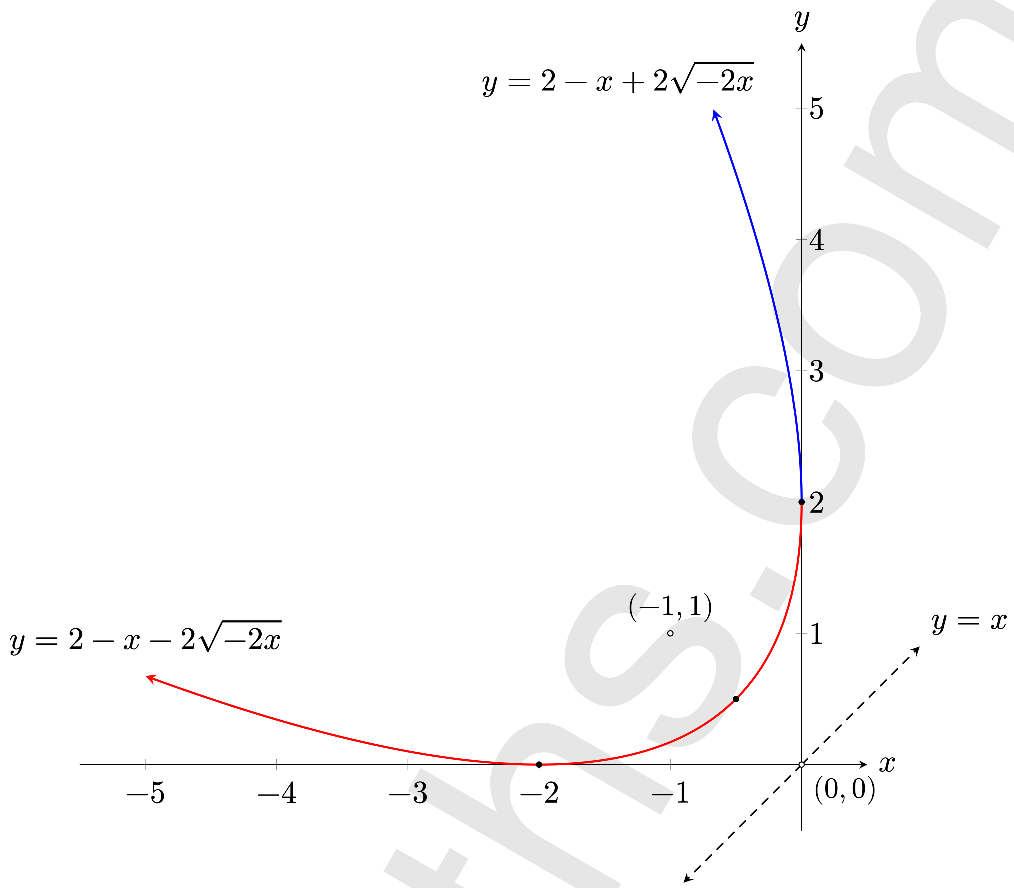
Markers note: The double root is not always sufficient, e.g. the curve $x^2 = y^2$, but the specifics are too hard for 2 marks only. A sketch of a proof: The equation of the locus can be factorised as $(x_p + y_p + 2)^2 = 8y_p$. For $y_p < 0$ there are no solutions for x_p, so along with the double root at $y_p = 0$ we conclude C is tangent to the x–axis.

(iii) By sketching C, or otherwise, find the area C encloses with the coordinate axis. **4**

It is recommended that you attempt to sketch C, as you will need an equation to integrate and a domain to integrate over. We solve for y_p in terms of x_p in the equation from (i).

$$\begin{aligned}
(x_p + 1)^2 + (y_p - 1)^2 &= \frac{(x_p - y_p)^2}{2}, \\
2(x_p^2 + 2x_p + 1) + 2(y_p^2 - 2y_p + 1) &= x_p^2 - 2x_p y_p + y^2, \\
x_p^2 + 4x_p + y_p^2 - 4y_p + 2x_p y_p + 4 &= 0, \\
y_p^2 + y_p(2x_p - 4) + (x_p + 2)^2 &= 0, \\
y_p &= \frac{4 - 2x_p \pm \sqrt{(2x_p - 4)^2 - 4(x_p + 2)^2}}{2} \\
&= 2 - x_p \pm \sqrt{(x_p - 2)^2 - (x_p + 2)^2} \\
&= 2 - x_p \pm 2\sqrt{-2x_p}.
\end{aligned}$$

From this and (ii) you should be able to get a decent idea of what C looks like.



To find the area, we integrate the lower branch of the locus from -2 to 0.

$$\begin{aligned}
A &= \int_{-2}^{0} \left(2 - x - 2\sqrt{-2x}\right) dx \\
&= \left[2x - \frac{x^2}{2} + \frac{2(-2x)^{\frac{3}{2}}}{3}\right]_{-2}^{0} \\
&= 0 - \left(-4 - \frac{4}{2} + \frac{2(4)^{\frac{3}{2}}}{3}\right) \\
&= \frac{2}{3}\,\mathrm{u}^2.
\end{aligned}$$

The integral can be confusing due to the presence of negatives in square roots, so a substitution $u = -x$ is also fine.

$$A = \int_{-2}^{0} \left(2 - x - 2\sqrt{-2x}\right) dx.$$

$$\text{Let } \begin{cases} u = -x, & \text{when } x = 0, u = 0, \\ du = -dx, & \text{when } x = -2, u = 2. \end{cases}$$

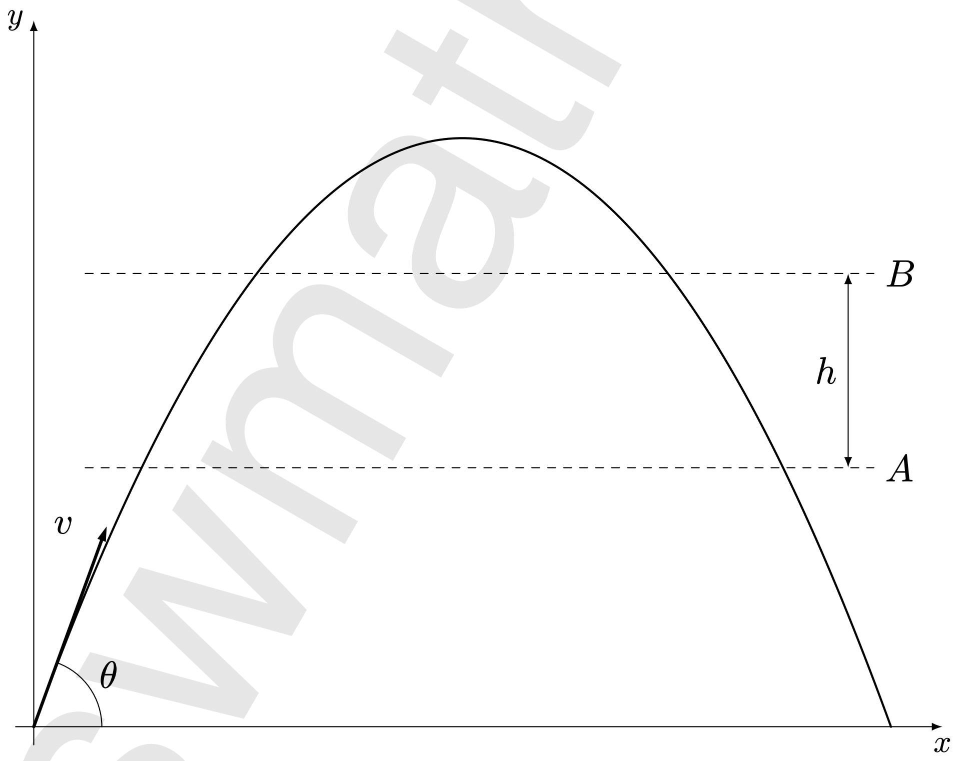
$$\therefore A = -\int_{2}^{0} \left(2 + u - 2\sqrt{2u}\right) du$$

$$= \int_0^2 \left(2 + u - 2\sqrt{2}u^{\frac{1}{2}}\right) du$$
$$= \left[2u + \frac{u^2}{2} - \frac{4\sqrt{2}u^{\frac{3}{2}}}{3}\right]_0^2$$
$$= \left(4 + \frac{4}{2} - \frac{4\sqrt{2} \times 2^{\frac{3}{2}}}{3}\right) - 0$$
$$= \frac{2}{3}\,\mathrm{u}^2.$$

(c) A projectile P is projected at an angle θ to the positive x-axis, at an initial velocity v. You may assume that the equation of motion for the projectile is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} vt\cos\theta \\ vt\sin\theta - \dfrac{gt^2}{2} \end{pmatrix}, \qquad \text{(Do NOT prove this.)}$$

where g is the acceleration of gravity.



Lines $y = A$ and $y = B$ have a vertical separation of h. The time between Ps two crossings of line A is recorded to be T_A. The same relation is true for line B and T_B.

(i) If $\alpha > \beta$ are the real two roots of $bx - ax^2 = c$, find an expression for $\alpha - \beta$. **1**

From the quadratic formula,

$$\alpha = \frac{b + \sqrt{b^2 - 4ac}}{2a}, \beta = \frac{b - \sqrt{b^2 - 4ac}}{2a},$$

so $\alpha - \beta = \dfrac{\sqrt{b^2 - 4ac}}{a}$.

(ii) Show that $g = \dfrac{8h}{T_A^2 - T_B^2}$. **3**

Consider the equation $vt \sin\theta - \dfrac{gt^2}{2} = A$. The two solutions to this equation are the times at which the projectile has a vertical displacement of A, so T_A must be equal to the difference of roots, which is given by

$$T_A = \frac{\sqrt{v^2 \sin^2\theta - 4(\frac{g}{2})(A)}}{\frac{g}{2}} = \frac{2\sqrt{v^2 \sin^2\theta - 2gA}}{g}$$

from (i). Similarly,

$$T_B = \frac{2\sqrt{v^2 \sin^2\theta - 2gB}}{g}.$$

Then,

$$\begin{aligned} T_A^2 - T_B^2 &= \frac{4(v^2 \sin^2\theta - 2gA) - 4(v^2 \sin^2\theta - 2gB)}{g^2} \\ &= \frac{8(B - A)}{g}, \\ g &= \frac{8(B - A)}{T_A^2 - T_B^2} \\ &= \frac{8h}{T_A^2 - T_B^2}. \end{aligned}$$

End of Question 13

Question 14 (16 marks) Use the Question 14 Writing Booklet

(a) (i) Solve the inequality $\dfrac{\sqrt{x+1}}{\sqrt{x}-\sqrt{x-1}} > x$ for x. **1**

Notice $\dfrac{\sqrt{x+1}}{\sqrt{x}-\sqrt{x-1}} = \dfrac{\sqrt{x+1}(\sqrt{x}+\sqrt{x-1})}{x-(x-1)} = \sqrt{x^2+x}+\sqrt{x^2-1}.$

But if $x > 0$, then $x^2+x > x^2$ so $\sqrt{x^2+x}+\sqrt{x^2-1} \geq \sqrt{x^2+x} > x$. Thus, when the LHS is defined (which is for all $x \geq 1$), the inequality must hold.

(ii) Prove by mathematical induction that if n is a positive integer, then **3**

$$\frac{\sqrt{2}}{1}+\frac{\sqrt{3}}{2}+\frac{\sqrt{4}}{3}+\cdots+\frac{\sqrt{n+1}}{n} > \sqrt{n}.$$

Base case: When $n = 1$, the LHS is $\sqrt{2}$ and the RHS is 1, so LHS $>$ RHS and the statement holds.

Inductive step: Assume the statement holds for $n = k \in \mathbb{Z}$, that is

$$\frac{\sqrt{2}}{1}+\frac{\sqrt{3}}{2}+\frac{\sqrt{4}}{3}+\cdots+\frac{\sqrt{k+1}}{k} > \sqrt{k},$$

we wish to prove the statement holds for $n = k+1$, that is

$$\frac{\sqrt{2}}{1}+\frac{\sqrt{3}}{2}+\frac{\sqrt{4}}{3}+\cdots+\frac{\sqrt{k+1}}{k}+\frac{\sqrt{k+2}}{k+1} > \sqrt{k+1},$$

We use the assumption immediately, to find that LHS $> \sqrt{k}+\dfrac{\sqrt{k+2}}{k+1}$. We would like to prove that this is $> \sqrt{k+1} =$ RHS, i.e.

$$\sqrt{k}+\frac{\sqrt{k+2}}{k+1} > \sqrt{k+1} \iff \frac{\sqrt{k+2}}{k+1} > \sqrt{k+1}-\sqrt{k},$$

and this looks awfully similar to the inequality in (i). In fact, if we let $x = k+1 \geq 1$,

$$\frac{\sqrt{k+2}}{\sqrt{k+1}-\sqrt{k}} > k+1 \implies \frac{\sqrt{k+2}}{k+1} > \sqrt{k+1}-\sqrt{k},$$

from (i) (worth mentioning $\sqrt{k+1}-\sqrt{k}, k+1 > 0$ so we can multiply and divide). Hence, LHS $>$ RHS so the statement holds for $n = k+1$ if it holds for $n = k$.

Conclusion: If you were expecting one, we don't do them here.

(iii) By considering the graph of $y = \dfrac{\sqrt{x}}{x}$ for $x \geq 1$, prove without mathematical induction that if n is a positive integer, then **3**

$$\frac{\sqrt{2}}{1} + \frac{\sqrt{3}}{2} + \frac{\sqrt{4}}{3} + \cdots + \frac{\sqrt{n+1}}{n} > 2\sqrt{n+1} - 2.$$

For each term in the sum, notice: $\dfrac{\sqrt{k+1}}{k} > \dfrac{\sqrt{k}}{k} = \dfrac{1}{\sqrt{k}}$.

Applying this to the entire sum, we obtain the inequality

$$\sum_{k=1}^{n} \frac{\sqrt{k+1}}{k} > \sum_{k=1}^{n} \frac{1}{\sqrt{k}}.$$

By drawing a graph, you can see that,

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \int_1^{n+1} \frac{1}{\sqrt{x}}\, dx,$$

from an approximation of the integral with n upper rectangles. We can evaluate the definite integral:

$$\begin{aligned} \int_1^{n+1} \frac{1}{\sqrt{x}}\, dx &= \int_1^{n+1} x^{-1/2}\, dx \\ &= 2\sqrt{n+1} - 2\sqrt{1} \\ &= 2\sqrt{n+1} - 2 \end{aligned}$$

And combining our inequalities, we have:

$$\sum_{k=1}^{n} \frac{\sqrt{k+1}}{k} > \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \int_1^{n+1} \frac{1}{\sqrt{x}}\, dx = 2\sqrt{n+1} - 2,$$

which is exactly what we want.

(b) An object is initially at rest A units away from a point P. It moves towards P due to gravitational acceleration that is inversely proportional to the square of the distance x of the object from P. Define positive to be away from P.

When the particle is x units away from P, it travels with speed v units towards P.

(i) Prove that **2**

$$v = -\sqrt{\frac{2k(A-x)}{Ax}}$$

where k is some constant of proportionality.

The question tells us that $a \propto -\frac{1}{x^2}$, so $a = -\frac{k}{x^2}$, where k is some positive constant.

$$\begin{aligned}
a &= -\frac{k}{x^2}, \\
v\frac{dv}{dx} &= -\frac{k}{x^2}, \\
\int_0^v v\,dv &= -k\int_A^x \frac{dx}{x^2}, \\
\frac{v^2}{2} &= k\left[\frac{1}{x} - \frac{1}{A}\right], \\
v^2 &= \frac{2k(A-x)}{Ax}.
\end{aligned}$$

Since the object is initially at rest A units away, and the only acceleration is one towards P, the object must always have nonpositive velocity. Thus, we take the negative square root, so

$$v = -\sqrt{\frac{2k(A-x)}{Ax}}.$$

(ii) Let T be the number of seconds it takes for the object to reach the half-way point between its initial position and P. **4**

Show that $T = \dfrac{A\sqrt{A}(\pi+2)}{4\sqrt{2k}}$.

Note halfway point is $x = A/2$. We want to find the time in seconds it takes for the object to reach this position.

$$\begin{aligned}
v &= -\sqrt{\frac{2k(A-x)}{Ax}} \text{ from (i)},\\
\frac{dx}{dt} &= -\sqrt{\frac{2k(A-x)}{Ax}},\\
\int_{\frac{A}{2}}^{A} \sqrt{\frac{x}{A-x}}\,dx &= \sqrt{\frac{2k}{A}}\int_0^T dt,\\
\int_{\frac{A}{2}}^{A} \frac{x}{\sqrt{x(A-x)}}\,dx &= \sqrt{\frac{2k}{A}}T,\\
\therefore T &= \sqrt{\frac{A}{2k}}\int_{\frac{A}{2}}^{A} \frac{x}{\sqrt{x(A-x)}}\,dx.
\end{aligned}$$

We solve the integral.

$$\begin{aligned}
\int_{\frac{A}{2}}^{A} \frac{x}{\sqrt{x(A-x)}}\,dx &= -\frac{1}{2}\int_{\frac{A}{2}}^{A} \frac{A-2x-A}{\sqrt{Ax-x^2}}\,dx\\
&= \frac{A}{2}\int_{\frac{A}{2}}^{A} \frac{dx}{\sqrt{Ax-x^2}} - \frac{1}{2}\int_{\frac{A}{2}}^{A} \frac{A-2x}{\sqrt{Ax-x^2}}\,dx,\\
&= \frac{A}{2}\int_{\frac{A}{2}}^{A} \frac{dx}{\sqrt{\frac{A^2}{4}-(x-\frac{A}{2})^2}} - \left[\sqrt{Ax-x^2}\right]_{\frac{A}{2}}^{A},\\
&= \frac{A}{2}\sin^{-1}\left(\frac{x-\frac{A}{2}}{\frac{A}{2}}\right)\Bigg|_{\frac{A}{2}}^{A} + \frac{A}{2}\\
&= \frac{A}{2}\left(\frac{\pi+2}{2}\right).\\
\therefore T &= \sqrt{\frac{A}{2k}}\cdot\frac{A}{2}\left(\frac{\pi+2}{2}\right)\\
T &= \frac{A\sqrt{A}(\pi+2)}{4\sqrt{2k}}.
\end{aligned}$$

(c) Using the substitution $u = x\sqrt{x}$, evaluate $\displaystyle\int_0^1 \sqrt{\frac{3x}{3+x^3}}\,dx$. **3**

Let $u = x\sqrt{x}$. Then, $du = \frac{3}{2}\sqrt{x}\,dx$. The bounds stay the same.

$$\begin{aligned}\int_0^1 \sqrt{\frac{3x}{3+x^3}}\,dx &= \frac{2}{3}\int_0^1 \sqrt{\frac{3}{3+u^2}}\,du,\\ &= \frac{2}{\sqrt{3}}\int_0^1 \frac{du}{\sqrt{3+u^2}},\end{aligned}$$

Let $u = \sqrt{3}\tan\theta$, $du = \sqrt{3}\sec^2\theta\,d\theta$. When $u = 0, \theta = 0$, and when $u = 1, \theta = \frac{\pi}{6}$.

$$\begin{aligned}&= \frac{2}{\sqrt{3}}\int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2\theta}{\sqrt{3+3\tan^2\theta}}\,d\theta\\ &= \frac{2}{\sqrt{3}}\int_0^{\frac{\pi}{6}} \sec\theta\,d\theta\\ &= \frac{2}{\sqrt{3}}\ln|\sec\theta + \tan\theta|\Big|_0^{\frac{\pi}{6}}\\ &= \frac{2}{\sqrt{3}}\ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\\ &= \frac{\ln\sqrt{3}}{2\sqrt{3}} = \frac{\ln 3}{\sqrt{3}}.\end{aligned}$$

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) Suppose $n \in \mathbb{Z}^-$. Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} (\sin x + 1)^n \, dx$.

(i) Show that **3**

$$I_n = I_{n+1} - 1 - n \int_0^{\frac{\pi}{2}} \cos^2 x (1 + \sin x)^{n-1} \, dx.$$

The intended method was to start by considering I_{n+1}:

$$\begin{aligned} I_{n+1} &= \int_0^{\frac{\pi}{2}} (1 + \sin x)(\sin x + 1)^n \, dx \\ &= \int_0^{\frac{\pi}{2}} (\sin x + 1)^n \, dx + \int_0^{\frac{\pi}{2}} \sin x (\sin x + 1)^n \, dx. \end{aligned}$$

The first integral is I_n, and we like that. The second integral can be approached with integration by parts. Suppose

$$\begin{cases} u = (1 + \sin x)^n, & du = n \cos x (1 + \sin x)^{n-1} dx, \\ dv = \sin x, & v = -\cos x \, dx. \end{cases}$$

Then, we have

$$\begin{aligned} I_{n+1} &= I_n - \Big[\cos x (1 + \sin x)^n\Big]_0^{\frac{\pi}{2}} + n \int_0^{\frac{\pi}{2}} \cos^2 x (1 + \sin x)^{n-1} \, dx \\ &= I_n + 1 + n \int_0^{\frac{\pi}{2}} \cos^2 x (1 + \sin x)^{n-1} \, dx, \\ \therefore I_n &= I_{n+1} - 1 - n \int_0^{\frac{\pi}{2}} \cos^2 x (1 + \sin x)^{n-1} \, dx. \end{aligned}$$

If you started with I_n, you would have found a relation from I_n to I_{n-1} and then had to remap n to $n + 1$,

(ii) Hence, show that **3**

$$I_n = \frac{n+1}{2n+1} I_{n+1} - \frac{1}{2n+1}.$$

Notice that $1 - (u - 1)^2 = 2u - u^2$. If we let $u = \sin x + 1$, we find

$$\cos^2 x = 1 - (\sin x + 1 - 1)^2 = 2(\sin x + 1) - (\sin x + 1)^2.$$

Thus,

$$\begin{aligned} I_n &= I_{n+1} - 1 - n \int_0^{\frac{\pi}{2}} \left[2(1 + \sin x) - (1 + \sin x)^2\right] (1 + \sin x)^{n-1} \, dx \\ &= I_{n+1} - 1 - 2n \int_0^{\frac{\pi}{2}} (1 + \sin x)^n \, dx + n \int_0^{\frac{\pi}{2}} (1 + \sin x)^{n+1} \\ &= (n + 1) I_{n+1} - 2n I_n - 1. \end{aligned}$$

Hence, $I_n = \dfrac{n+1}{2n+1} I_{n+1} - \dfrac{1}{2n+1}$.

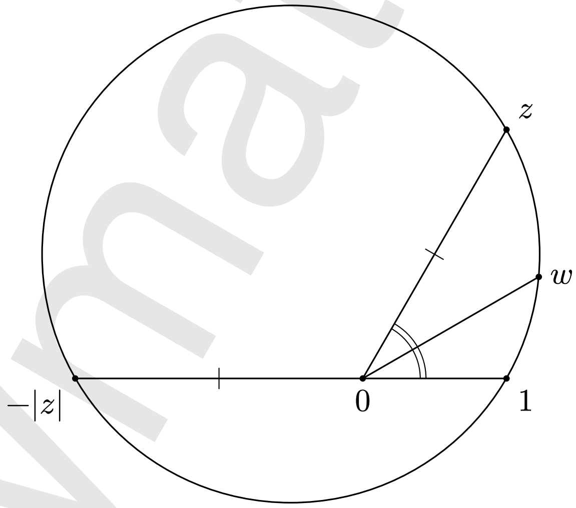
(iii) Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{(1+\sin x)^4} = \frac{12}{35}$. **1**

Very classic end of reduction question. We are asked to find I_{-4}. Note I_0 is technically undefined, however we are fine with you using this method, even if you did not find it. If you went ahead and calculated I_1 manually, that is cool too. Using the reduction,

$$
\begin{aligned}
I_{-1} &= \frac{-1+1}{-2+1} I_0 - \frac{1}{-2+1} = 1,\\
I_{-2} &= \frac{-2+1}{-4+1} I_1 - \frac{1}{-4+1} = \frac{2}{3},\\
I_{-3} &= \frac{-3+1}{-6+1} I_2 - \frac{1}{-6+1} = \frac{7}{15},\\
I_{-4} &= \frac{-4+1}{-8+1} I_3 - \frac{1}{-8+1} = \frac{12}{35}.
\end{aligned}
$$

(b) Let z and w be two complex numbers in the first quadrant of the Argand diagram, with $\operatorname{Arg}(z) = 2\operatorname{Arg}(w)$. It is known that there exists a circle which passes through the four complex numbers $1, z, w$, and $-|z|$.

This information is shown in the following diagram.



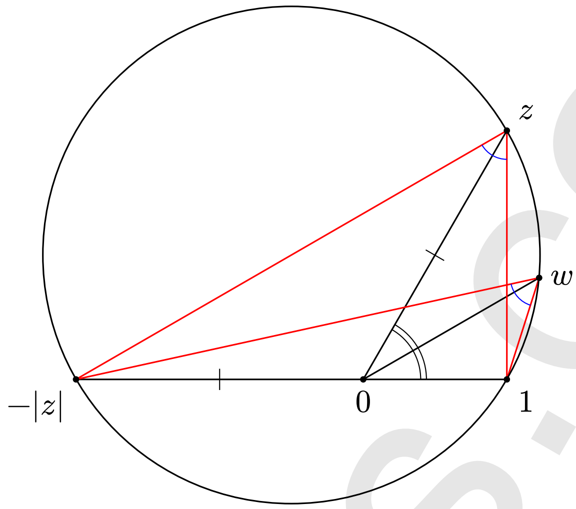
(i) Explain why there is a positive real number k satisfying $kw^2 = z$. **1**

Since w is in the first quadrant, $2\operatorname{Arg}(w) < \pi$, hence $\operatorname{Arg}(w^2) = 2\operatorname{Arg}(w)$. Then, z and w^2 have the same argument so they lie on the same ray from the origin, hence k exists. Or say $\dfrac{z}{w^2}$ must be real.

(ii) By considering two equal angles subtended by the same arc, prove that **3**

$$\frac{(z-1)(w+|z|)}{(w-1)(z+|z|)}$$

is purely real.



Notice that the two blue angles are subtended by the same arc and thus are equal, so

$$\operatorname{Arg}\left(\frac{z-1}{z+|z|}\right) = \operatorname{Arg}\left(\frac{w-1}{w+|z|}\right),$$

So they must lie on the same ray from the origin, by the same reasoning as (i) we find

$$\frac{z-1}{z+|z|} \bigg/ \frac{w-1}{w+|z|} = \frac{(z-1)(w+|z|)}{(w-1)(z+|z|)} \in \mathbb{R}.$$

(iii) Deduce that w is one of the square roots of z. **4**

Recall that if z is real then $z = \bar{z}$. Thus,

$$\begin{aligned}
\frac{(z-1)(w+|z|)}{(w-1)(z+|z|)} &= \overline{\frac{(z-1)(w+|z|)}{(w-1)(z+|z|)}} \\
&= \frac{(\bar{z}-1)(\bar{w}+|z|)}{(\bar{w}-1)(\bar{z}+|z|)}, \\
\frac{(kw^2-1)(w+k|w|^2)}{(w-1)(kw^2+k|w|^2)} &= \frac{(k\bar{w}^2-1)(\bar{w}+k|w|^2)}{(\bar{w}-1)(k\bar{w}^2+k|w|^2)}, \\
\frac{(kw^2-1)(w+kw\bar{w})}{(w-1)(w^2+w\bar{w})} &= \frac{(k\bar{w}^2-1)(\bar{w}+kw\bar{w})}{(\bar{w}-1)(\bar{w}^2+w\bar{w})}, \\
\frac{(kw^2-1)(1+k\bar{w})}{(w-1)(w+\bar{w})} &= \frac{(k\bar{w}^2-1)(1+kw)}{(\bar{w}-1)(w+\bar{w})},
\end{aligned}$$

At this point, we are pushed to clear denominators and expand.
Note $w + \bar{w} = 2\operatorname{Re}(w) > 0$ as w is in the first quadrant, so we can cancel it.

$$
\begin{aligned}
(kw^2-1)(1+k\bar{w})(\bar{w}-1) &= (k\bar{w}^2-1)(1+kw)(w-1),\\
(kw^2-1+k^2w^2\bar{w}-k\bar{w})(\bar{w}-1) &= (k\bar{w}^2-1+k^2w\bar{w}^2-kw)(w-1),\\
kw^2\bar{w}-\bar{w}+\cancel{k^2w^2\bar{w}^2}-\cancel{k\bar{w}^2}-\cancel{kw^2}+\cancel{1}-k^2w^2\bar{w}+k\bar{w} &= kw\bar{w}^2-w+\cancel{k^2w^2\bar{w}^2}-\cancel{kw^2}-\cancel{k\bar{w}^2}+\cancel{1}-k^2w\bar{w}^2+kw,\\
kw^2\bar{w}-\bar{w}-k^2w^2\bar{w}+k\bar{w} &= kw\bar{w}^2-w-k^2w\bar{w}^2+kw,\\
0 &= kw\bar{w}(\bar{w}-w)+(\bar{w}-w)+k^2w\bar{w}(w-\bar{w})+k(w-\bar{w})\\
&= (w-\bar{w})(k^2|w|^2+k-k|w|^2-1)\\
&= 2i\,\mathrm{Im}(w)(k|w|^2+1)(k-1).
\end{aligned}
$$

Since w is in the first quadrant, $\mathrm{Im}(w)\neq 0$. Since k is a positive real number, $k|w|^2+1\neq 0$. Thus, $k=1$ so $z=w^2$, i.e. w is a square root of z.

Since there is some space here, I (MagiicMushroom) would like to talk about the design of the paper. Do take into account that most of the questions were written and placed on the day of the exam, from around 2AM to 8:30AM. I wrote around 200 marks of stuff and when 187 woke up, we worked through what was there and built this test. The rest of it will be released for fun.

Out of MC, the only notable ones are Q2 and Q10. Q2 was intended as a jumpscare, especially in reading time. Students who wasted time trying to do it in their head generally made more mistakes in the rest of MC. Once Q2 had knocked you out of your comfort zone, Q10 had a bait answer and was also very hard for a MC.

Q11 was standard, (b) was inspired by a Ruiace Q. Q12 (a) is my favourite question of the exam, the amount of people who tried to bs it was funny, because people are conditioned to think of inequalities as this hard topic requiring lots of ingenuity and random asspulls. Granted, the solution is not trivial, but it is far from what people were trying to do.

Q13 (c) and Q14 (b) had typos in them when we printed the exams, apologies to all who got confused. But up to (not including) Q15, the exam had nothing requiring more than normal amounts of insight. Most questions where I used a trick, I present alternate solutions.

Q15 was intended to be at the level of a normal HSC Q16. It was also meant to be quite long and tedious, to stress you out both mentally and with regard to time going into Q16. The first part of Q16 is much easier than the whole of Q15, however it was just as poorly attempted as Q15 due to the pressure most likely. And the end of Q16...

If you find typos or mistakes in any of these solutions, please DM magiicmushroom on discord.

进入决赛!

End of Question 15

Question 16 (14 marks) Use the Question 16 Writing Booklet

(a) Suppose $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$ are nonzero 3D vectors satisfying the following three conditions:

1. $|\underset{\sim}{a}+\underset{\sim}{b}| = |\underset{\sim}{a}-\underset{\sim}{b}| = |\underset{\sim}{c}|$,
2. $|\underset{\sim}{a}| = 3|\underset{\sim}{b}|$, and
3. $(2\underset{\sim}{a}+\underset{\sim}{b})$ is parallel to $\underset{\sim}{c}$.

(i) Show that $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$. **1**

Expand $|\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}-\underset{\sim}{b}|^2$ into $|\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2$ so $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$.

(ii) Explain why there exists a real number λ satisfying $2\underset{\sim}{a}+\underset{\sim}{b} = \lambda\underset{\sim}{c}$, and show further that λ satisfies **2**

$$\lambda^2 = \frac{4|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2}{|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2}.$$

We are given $(2\underset{\sim}{a}+\underset{\sim}{b})$ is parallel to $\underset{\sim}{c}$, it is immediate that such λ exists. Taking magnitudes squared,

$$\begin{aligned}
|2\underset{\sim}{a}+\underset{\sim}{b}|^2 &= \lambda^2|\underset{\sim}{c}|^2,\\
4|\underset{\sim}{a}|^2 + 4\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2 &= \lambda^2|\underset{\sim}{a}+\underset{\sim}{b}|^2,\\
4|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 &= \lambda^2(|\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2)\\
&= \lambda^2(|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2),\\
\lambda^2 &= \frac{4|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2}{|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2}.
\end{aligned}$$

(iii) Find all possible values of $\underset{\sim}{c}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **2**

Substitute $|\underset{\sim}{a}| = 3|\underset{\sim}{b}|$ into (ii):

$$\begin{aligned}
\lambda^2 &= \frac{4(9|\underset{\sim}{b}|^2) + |\underset{\sim}{b}|^2}{9|\underset{\sim}{b}|^2 + |\underset{\sim}{b}|^2}\\
&= \frac{37}{10},\\
\lambda &= \pm\sqrt{\frac{37}{10}}.
\end{aligned}$$

Thus, $\underset{\sim}{c} = \dfrac{1}{\lambda}(2\underset{\sim}{a}+\underset{\sim}{b}) = \pm\sqrt{\dfrac{10}{37}}(2\underset{\sim}{a}+\underset{\sim}{b})$.

(b) Suppose $n \geq 2$ is a positive integer, and let $\omega = e^{\frac{\pi i}{n}}$ be an $2n$th root of unity. Consider the polynomial $P(z) = z^{2n} - 1$.

(i) Show that $2n = \left|(1-\omega)(1-\omega^2)\cdots(1-\omega^{2n-1})\right|$. **2**

Note that $z^{2n} - 1 \equiv (z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{2n-1})$.
Also, $z^{2n} - 1 \equiv (z-1)(z^{2n-1} + z^{2n-2} + \cdots + 1)$. Dividing out the factor of $z - 1$,

$$\begin{aligned} z^{2n-1} + z^{2n-2} + \cdots + 1 &= (z-\omega)(z-\omega^2)\cdots(z-\omega^{2n-1}), \\ \underbrace{1 + 1 + \cdots + 1}_{2n \text{ 1s}} &= (1-\omega)(1-\omega^2)\cdots(1-\omega^{2n-1}), \\ 2n &= (1-\omega)(1-\omega^2)\cdots(1-\omega^{2n-1}), \\ |2n| = 2n &= |(1-\omega)(1-\omega^2)\cdots(1-\omega^{2n-1})|. \end{aligned}$$

(ii) Hence, show that $\displaystyle\sum_{k=1}^{n-1} \ln\left(\sin\frac{\pi k}{2n}\right) = \frac{1}{2}\ln n - (n-1)\ln 2$. **4**

First, we calculate:

$$\begin{aligned} |1-\omega^k| &= \left|1 - e^{\frac{k\pi i}{n}}\right| = \left|e^{\frac{k\pi i}{n}} - 1\right| \\ &= \left|e^{\frac{k\pi i}{2n}}\right|\left|e^{\frac{k\pi i}{2n}} - e^{-\frac{k\pi i}{2n}}\right| \\ &= 1 \cdot \left|2i\sin\frac{k\pi}{2n}\right| \\ &= 2\sin\frac{k\pi}{2n}, \end{aligned}$$

as $1 \leq k < 2n - 1$, so

$$\begin{aligned} |(1-\omega)(1-\omega^2)\cdots(1-\omega^{2n-1})| &= \left(2\sin\frac{\pi}{2n}\right)\left(2\sin\frac{2\pi}{2n}\right)\cdots\left(2\sin\frac{(2n-1)\pi}{2n}\right), \\ \ln(2n) &= \ln\left(2^{2n-1}\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\cdots\sin\frac{(2n-1)\pi}{2n}\right), \\ \ln 2 + \ln n - (2n-1)\ln 2 &= \ln\left(\sin\frac{\pi}{2n}\right) + \ln\left(\sin\frac{2\pi}{2n}\right) + \cdots + \ln\left(\sin\frac{(2n-1)\pi}{2n}\right). \end{aligned}$$

This looks close to what we need, but we have too many sin terms inside the lns! We can use a symmetry argument to fix this.

Recall that $\sin x = \sin(\pi - x)$. Consider the sum of $\ln \sin$ terms:

$$\ln\left(\sin\frac{\pi}{2n}\right) + \cdots + \ln\left(\sin\frac{(n-1)\pi}{2n}\right) + \ln\left(\sin\frac{n\pi}{2n}\right) + \ln\left(\sin\frac{(n+1)\pi}{2n}\right) + \cdots + \ln\left(\sin\frac{(2n-1)\pi}{2n}\right)$$
$$= \ln\left(\sin\frac{\pi}{2n}\right) + \cdots + \ln\left(\sin\frac{(n-1)\pi}{2n}\right) + \ln(1) + \ln\left(\sin\frac{2n\pi-(n+1)\pi}{2n}\right) + \cdots + \ln\left(\sin\frac{2n\pi-(2n-1)\pi}{2n}\right)$$
$$= \ln\left(\sin\frac{\pi}{2n}\right) + \cdots + \ln\left(\sin\frac{(n-1)\pi}{2n}\right) + \cdots + \ln\left(\sin\frac{(n-1)\pi}{2n}\right) + \cdots + \ln\left(\sin\frac{\pi}{2n}\right)$$
$$= 2\sum_{k=1}^{n-1} \ln\left(\sin\frac{\pi k}{2n}\right).$$

Hence,

$$\ln n - 2(n-1)\ln 2 = 2\sum_{k=1}^{n-1} \ln\left(\sin\frac{\pi k}{2n}\right),$$
$$\sum_{k=1}^{n-1} \ln\left(\sin\frac{\pi k}{2n}\right) = \frac{1}{2}\ln n - (n-1)\ln 2.$$

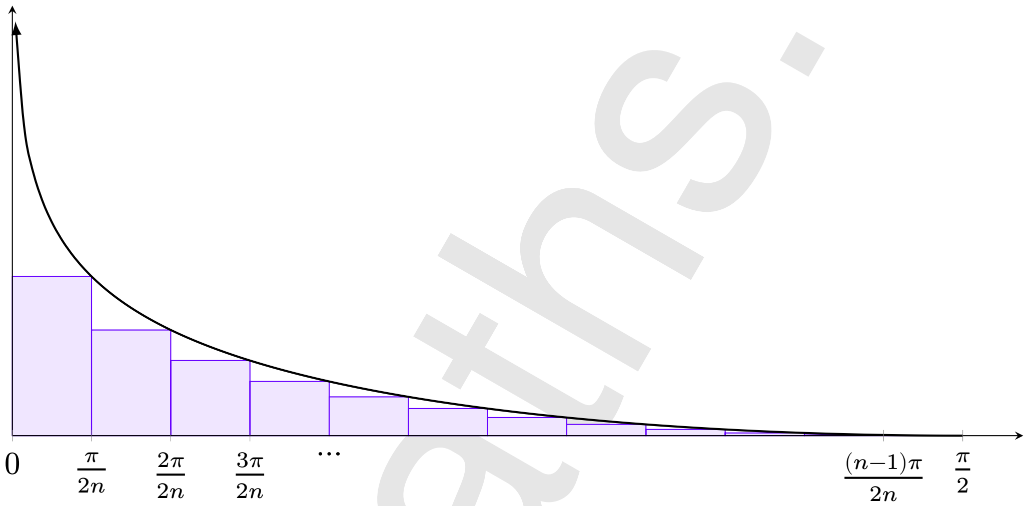
(c) Consider the function $f(x) = \ln(\csc x)$ for $x \in \left(0, \frac{\pi}{2}\right]$. **3**

You may assume that $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} = 0$ (Do NOT prove this).

Find an expression for S_n, the sum of the areas of n lower rectangles for $f(x)$ over the interval $\left(0, \frac{\pi}{2}\right]$, and hence conclude using (b) that

$$\int_0^{\frac{\pi}{2}} \ln(\csc x)\, dx = \frac{\pi}{2}\ln 2.$$

Consider splitting the interval $\left(0, \frac{\pi}{2}\right]$ into n intervals. Each interval must then have a length of $\frac{\pi}{2n}$. On each interval, construct a lower rectangle. Note that since $\sin x$ is an increasing function on $\left[0, \frac{\pi}{2}\right]$, $\csc x$ is a decreasing function, and so is $\ln(\csc x)$.



The total area of these n lower rectangles is given by

$$S_n = \frac{\pi}{2n}\sum_{k=1}^{n} \ln\left(\csc\frac{\pi k}{2n}\right) = -\frac{\pi}{2n}\sum_{k=1}^{n-1} \ln\left(\sin\frac{\pi k}{2n}\right) = \frac{\pi}{2n}\left((n-1)\ln 2 - \frac{1}{2}\ln n\right),$$

using (b), noting the nth rectangle has zero area so we can get rid of the $n = k$ term in the sum, and since $\displaystyle\lim_{n\to\infty} S_n = \int_0^{\pi} \ln(\csc x)\, dx$, we compute

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{\pi}{2n}\left((n-1)\ln 2 - \frac{1}{2}\ln n\right) \\
&= \frac{\pi \ln 2}{2}\lim_{n\to\infty}\frac{n-1}{n} - \frac{\pi}{4}\lim_{n\to\infty}\frac{\ln n}{n} \\
&= \frac{\pi\ln 2}{2}\cdot 1 - 0, \\
\therefore \int_0^{\frac{\pi}{2}} \ln(\csc x)\, dx &= \frac{\pi}{2}\ln 2.
\end{aligned}$$

End of paper

床前明月光
疑是地上霜
举头望明月
低头思故乡

李白, 静夜思.

SOLUTIONS

Justin	Q14a iii
MagiicMushroom	Everything else.

one more day till i go home

just one more day.

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

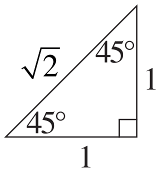
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

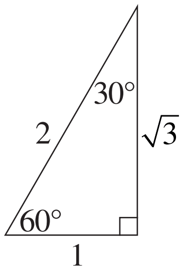
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

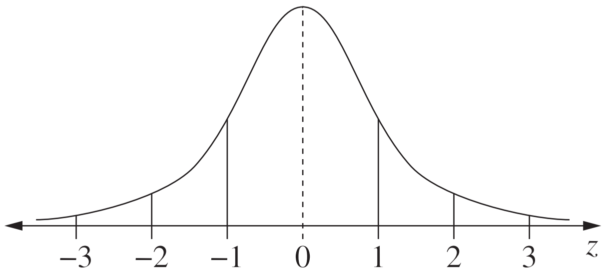
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)} dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)} dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}} dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2} dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx} dx = uv - \int v\frac{du}{dx} dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {^nC_r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2\,,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$