NSW GOVERNMENT

NSW Education Standards Authority

Centre Number

Student Number

This sample HSC examination paper is designed to show one way the Mathematics Extension 2 Year 12 Syllabus (2024) could be examined. It reflects the layout of a formatted HSC examination paper.

Sample **HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 100

Section I – 10 marks (pages 3–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 8–15)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

The first HSC examination for the Mathematics Extension 2 Year 12 Syllabus (2024) will be held in 2027. The Mathematics Extension 2 examination does not include items that are common with other Mathematics HSC examinations. Where they are still relevant, past HSC questions have been used to illustrate different types of questions and how the syllabus can be examined.

The HSC Mathematics Extension 2 examination specifications can be found in the assessment information for the Mathematics Extension 2 Year 12 Syllabus (2024).

This sample examination provides examples of some types of questions that may be found in HSC examinations for Mathematics Extension 2. Each question has been mapped to show how the question relates to syllabus outcomes and content.

Answers for the objective-response questions (Section I) and sample answers and marking guidelines for all other questions (Section II) are provided. The marking guidelines indicate the criteria for each mark.

In the examination, students will record their answers to Section I on a multiple-choice answer sheet and their answers to Section II in the appropriate writing booklets.

The sample questions, marking criteria, sample answers and annotations provide teachers and students with guidance as to the types of questions to expect and how they may be marked. They are not meant to be prescriptive. Each year the structure of the examination may differ in the number and types of questions, or focus on different syllabus outcomes and content.

Note:

Comments in coloured boxes are annotations that provide information and guidance.

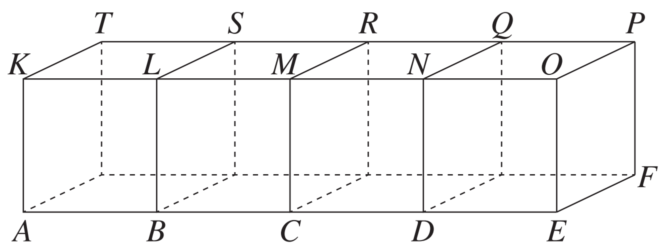
Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

Past HSC examinations provide other examples of questions that may be asked in this section.

1 Four cubes are placed in a line as shown on the diagram.



Which of the following vectors is equal to $\overrightarrow{AB} + \overrightarrow{CQ}$?

A. $\overrightarrow{AQ}$

B. $\overrightarrow{CP}$

C. $\overrightarrow{PB}$

D. $\overrightarrow{RA}$

2 Which expression is equal to $\int x^5 e^{7x}\,dx$?

A. $\frac{1}{7}x^5e^{7x} - \frac{5}{7}\int x^4e^{7x}\,dx$

B. $\frac{1}{7}x^5e^{7x} - \frac{5}{7}\int x^5e^{7x}\,dx$

C. $\frac{5}{7}x^4e^{7x} - \frac{5}{7}\int x^4e^{7x}\,dx$

D. $\frac{5}{7}x^4e^{7x} - \frac{5}{7}\int x^5e^{7x}\,dx$

3 Which of the following is a vector equation of the line joining the points $A(4, 2, 5)$ and $B(-2, 2, 1)$?

A. $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

B. $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

C. $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$

D. $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$

4 Consider the statement:

'For all integers n, if n is a multiple of 6, then n is a multiple of 2'.

Which of the following is the contrapositive of the statement?

A. There exists an integer n such that n is a multiple of 6 and not a multiple of 2.

B. There exists an integer n such that n is a multiple of 2 and not a multiple of 6.

C. For all integers n, if n is not a multiple of 2, then n is not a multiple of 6.

D. For all integers n, if n is not a multiple of 6, then n is not a multiple of 2.

5 Which of the following statements is FALSE?

A. $\forall\, a, b \in \mathbb{R}, \quad a < b \;\Rightarrow\; a^3 < b^3$

B. $\forall\, a, b \in \mathbb{R}, \quad a < b \;\Rightarrow\; e^{-a} > e^{-b}$

C. $\forall\, a, b \in (0, \infty), \quad a < b \;\Rightarrow\; \ln a < \ln b$

D. $\forall\, a, b \in \mathbb{R}$, with $a, b \neq 0, \quad a < b \;\Rightarrow\; \dfrac{1}{a} > \dfrac{1}{b}$

6 The complex number z satisfies $|z - 1| = 1$.

What is the greatest distance that z can be from the point i on the complex plane?

A. 1

B. $\sqrt{5}$

C. $2\sqrt{2}$

D. $\sqrt{2}+1$

7 Which polynomial could have $2 + i$ as a zero, given that k is a real number?

A. $x^3 - 4x^2 + kx$

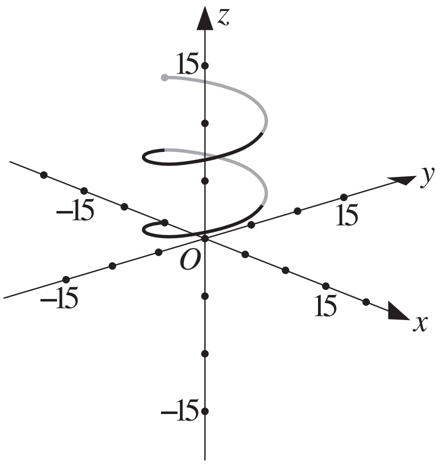
B. $x^3 - 4x^2 + kx + 5$

C. $x^3 - 5x^2 + kx$

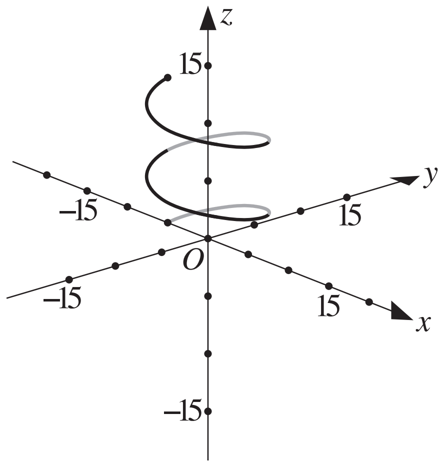
D. $x^3 - 5x^2 + kx + 5$

8 Which diagram best shows the curve described by the position vector $\mathbf{r}(t) = -5\cos(t)\mathbf{i} + 5\sin(t)\mathbf{j} + t\mathbf{k}$ for $0 \le t \le 4\pi$?

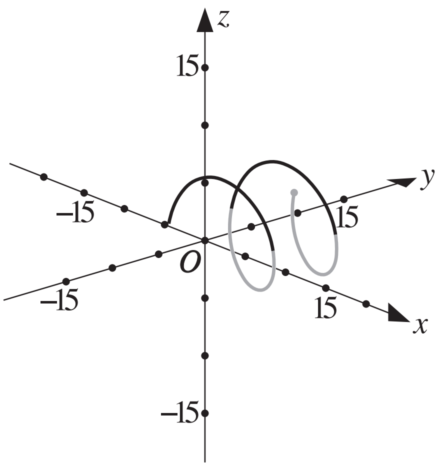
A.



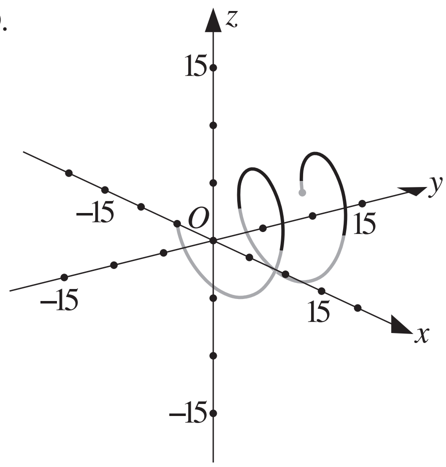
B.



C.



D.



9 Consider the proposition:

$$\text{`If } 2^n - 1 \text{ is not prime, then } n \text{ is not prime'.}$$

Given that each of the following statements is true, which statement disproves the proposition?

A. $2^5 - 1$ is prime

B. $2^6 - 1$ is divisible by 9

C. $2^7 - 1$ is prime

D. $2^{11} - 1$ is divisible by 23

10 A particle is moving vertically in a resistive medium under the influence of gravity. The resistive force is proportional to the velocity of the particle.

The initial speed of the particle is NOT zero.

Which of the following statements about the motion of the particle is always true?

A. If the particle is initially moving downwards, then its speed will increase.

B. If the particle is initially moving downwards, then its speed will decrease.

C. If the particle is initially moving upwards, then its speed will eventually approach a terminal speed.

D. If the particle is initially moving upwards, then its speed will not eventually approach a terminal speed.

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Past HSC examinations provide other examples of questions that may be asked in this section.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) The complex numbers $z = 5 + i$ and $w = 2 - 4i$ are given. **2**

Find $\dfrac{\overline{z}}{w}$, giving your answer in Cartesian form.

(b) Express $\dfrac{3x^2 - 5}{(x - 2)(x^2 + x + 1)}$ as a sum of partial fractions over $\mathbb{R}$. **3**

(c) (i) Find the two square roots of $-i$, giving the answers in the form $x + iy$, where x and y are real numbers. **2**

(ii) Hence, or otherwise, solve $z^2 + 2z + 1 + i = 0$ giving your solutions in the form $a + ib$ where a and b are real numbers. **2**

Question 11 continues on page 9

(d) Consider the triangle with vertices $A(4, 5, 1)$, $B(8, 1, 2)$ and the origin $O(0, 0, 0)$. The triangle has three medians.

The median through B has vector equation

$$\lambda\,\overrightarrow{OM} + (1-\lambda)\,\overrightarrow{OB} = \begin{pmatrix} 8 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 1.5 \\ -1.5 \end{pmatrix} \qquad \text{(Do NOT prove this.)}$$

for a parameter $\lambda \in [0, 1]$ and where M is the midpoint of OA.

(i) Write down the median through A as a vector equation. **2**

(ii) The three medians of any triangle meet at a point known as the centroid. **2**

Find the value of λ corresponding to the centroid.

(iii) Into what ratio does the centroid divide the median BM? **1**

(e) Consider Statement A.

Statement A: 'If n^2 is even, then n is even.'

(i) What is the converse of Statement A? **1**

(ii) Show that the converse of Statement A is true. **1**

End of Question 11

Question 12 (16 marks) Use the Question 12 Writing Booklet

(a) Find $\displaystyle\int \frac{2x+3}{x^2+2x+2}\,dx$. **3**

(b) Line 1 is given by the equations $x = -1 + 2s$, $y = 1 - 2s$ and $z = 1 + 2s$, where s is a parameter. **3**

Line 2 is given by the equations $x = 1 + 2t$, $y = -1 - t$ and $z = 4 + 3t$, where t is a parameter.

Show that Line 1 and Line 2 are skew.

(c) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the plane, where $\mathbf{v} \neq \mathbf{0}$.

For every real number t, let $P(t) = \left| \mathbf{u} - t\mathbf{v} \right|^2$.

(i) Show that $P(t) = |\mathbf{u}|^2 - 2t(\mathbf{u} \cdot \mathbf{v}) + t^2|\mathbf{v}|^2$. **1**

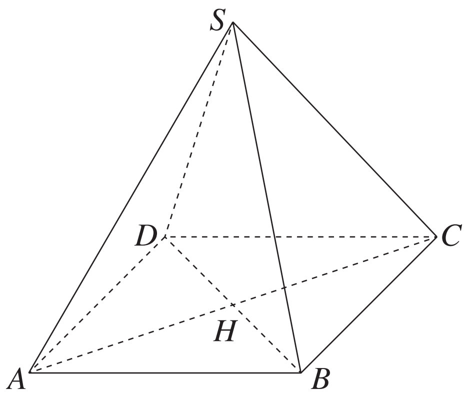
(ii) Show that $P(t)$ has a minimum value at $t = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}$. **2**

(iii) Hence prove the Cauchy–Schwarz inequality, $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$. **3**

Question 12 continues on page 11

Question 12 (continued)

(d) The diagram shows the pyramid $ABCDS$ where $ABCD$ is a square. The diagonals of the square bisect each other at H.



(i) Show that $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD} = \mathbf{0}$. **1**

Let G be the point such that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} + \overrightarrow{GS} = \mathbf{0}$.

(ii) Using part (i), or otherwise, show that $4\overrightarrow{GH} + \overrightarrow{GS} = \mathbf{0}$. **2**

(iii) Find the value of λ such that $\overrightarrow{HG} = \lambda\overrightarrow{HS}$. **1**

End of Question 12

Question 13 (14 marks) Use the Question 13 Writing Booklet

(a) The location of the complex number $a + ib$ is shown on the diagram on page 1 of the Question 13 Writing Booklet. **2**

On the diagram provided in the writing booklet, indicate the locations of all of the fourth roots of the complex number $a + ib$.

(b) An object is moving in simple harmonic motion along the x-axis. The acceleration of the object is given by $\ddot{x} = -4(x - 3)$ where x is its displacement from the origin, measured in metres, after t seconds.

Initially, the object is 5.5 metres to the right of the origin and moving towards the origin. The object has a speed of 8 m s^{-1} as it passes through the origin.

(i) Between which two values of x is the particle oscillating? **2**

(ii) Find the first value of t for which $x = 0$, giving the answer correct to 2 decimal places. **2**

(c) In a circus act, an 8 kg cannon ball is projected from the origin into the air with an initial velocity of 26 m s^{-1} and at an angle of 67.4° to the horizontal. The ball is caught at the top of its trajectory by a performer who is at the position (A, B).

The velocity vector, $\mathbf{v}(t)$, of the ball at time t seconds after launch is given by

$$\mathbf{v}(t) = 10e^{-0.8t}\mathbf{i} + \left[\,36.5\,e^{-0.8t} - 12.5\right]\mathbf{j}\ .$$ (Do NOT prove this.)

(i) Show that the ball reaches the performer at $t = 1.339$ (to three decimal places). **2**

(ii) Find the values of A and B. **3**

(iii) What is the speed of the ball when it reaches the performer? **1**

(iv) What is the magnitude of the force on the ball when it reaches the performer? **2**

Question 14 (14 marks) Use the Question 14 Writing Booklet

(a) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\,dx$. **4**

(b) An object of mass 5 kg is on a slope that is inclined at an angle of 60° to the horizontal. The acceleration due to gravity is g m s^{-2} and the velocity of the object down the slope is v m s^{-1}.

As well as the force due to gravity, the object is acted on by two forces, one of magnitude $2v$ newtons and one of magnitude $2v^2$ newtons, both acting up the slope.

(i) Show that the resultant force down the slope is **2**
$\dfrac{5\sqrt{3}}{2}g - 2v - 2v^2$ newtons.

(ii) There is one value of v such that the object will slide down the slope at a constant speed. **2**

Find this value of v in m s^{-1}, correct to 1 decimal place, given that $g = 10$.

(c) (i) Using de Moivre's theorem and the binomial expansion of $(\cos\theta + i\sin\theta)^5$, or otherwise, show that **3**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(ii) By using part (i), or otherwise, show that the real parts of the 5th roots of i are 0 and $\pm\sqrt{\dfrac{5+\sqrt{5}}{8}}$. **3**

Question 15 (14 marks) Use the Question 15 Writing Booklet

(a) Evaluate $\displaystyle\int_{\sqrt{10}}^{\sqrt{13}} x^3\sqrt{x^2-9}\,dx$. **3**

(b) An object of mass 1 kg is projected vertically upwards with an initial velocity of u m/s. It experiences air resistance of magnitude kv^2 newtons where v is the velocity of the object, in m/s, and k is a positive constant. The height of the object above its starting point is x metres. The time since projection is t seconds and acceleration due to gravity is g m/s^2.

(i) Show that the time for the object to reach its maximum height is $\dfrac{1}{\sqrt{gk}}\arctan\left(u\sqrt{\dfrac{k}{g}}\right)$ seconds. **3**

(ii) Find an expression for the maximum height reached by the object, in terms of k, g and u. **3**

(c) Let $I_n = \displaystyle\int_{-3}^{0} x^n\sqrt{x+3}\,dx$ for $n = 0, 1, 2, \ldots$

(i) Show that, for $n \geq 1$, **3**

$$I_n = \frac{-6n}{3+2n}I_{n-1}.$$

(ii) Find the value of I_2. **2**

Question 16 (16 marks) Use the Question 16 Writing Booklet

(a) On the complex plane, sketch the set of complex numbers $z = x + iy$ satisfying the equation $\text{Re}(z) = \text{Arg}(z)$. **3**

(b) A sphere of radius $r = 3$ is centred at $C(6, -3, 2)$. **4**

A line passes through $A(3, -1, 6)$ and $B(5, -1, -5)$.

Show that this line is a tangent to the sphere.

(c) Consider a sequence of rectangles with side lengths a_n and b_n.

The first rectangle has $a_1 = 2$ and $b_1 = 1$.

For integers $n \geq 1$,

$$a_{n+1} = \frac{a_n + b_n}{2} \text{ and } b_{n+1} = \frac{2}{a_{n+1}}.$$

(i) Show that every rectangle in the sequence has an area of 2 square units. **1**

(ii) Use the relationship between the arithmetic mean and the geometric mean to prove that $a_n \geq \sqrt{2}$ for any integer $n \geq 1$. **2**

(iii) Use mathematical induction to prove that $a_n - \sqrt{2} \leq \frac{1}{2^{n-1}}\left(2 - \sqrt{2}\right)$ for any integer $n \geq 1$. **4**

(iv) Use the squeeze theorem to show that the rectangles approach a square as n approaches infinity. **2**

End of paper

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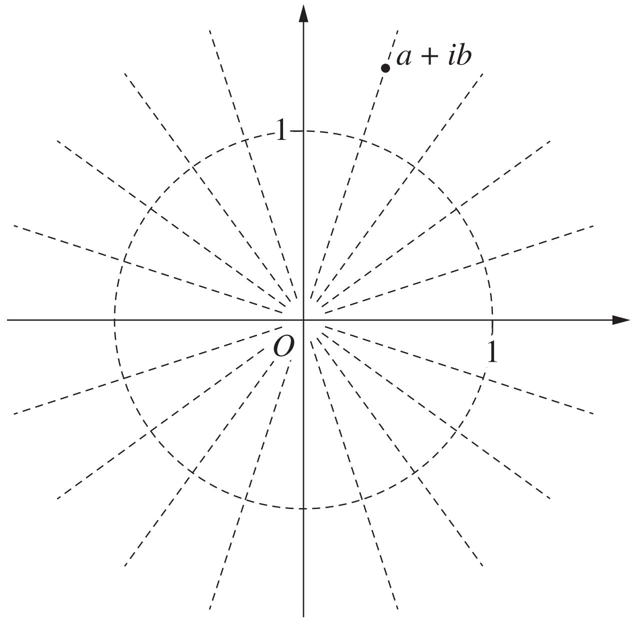
Start here for Question Number: **13**

(a) Indicate the locations of all of the fourth roots of the complex number $a + ib$.



This diagram has been included to illustrate how students would respond to 13 (a) in the designated writing booklet.

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Sample 2027 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

The reference sheet has been updated. Additions have been highlighted in red boxes below.

REFERENCE SHEET

Mathematics Extension 2

Trigonometric identities

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

Integral calculus

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Vectors

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

Complex numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = A\cos(nt+\alpha)+c$$

$$x = A\sin(nt+\alpha)+c$$

$$\ddot{x} = -n^2(x-c)$$

$$\boxed{v^2 = n^2\left[A^2 - (x-c)^2\right]}$$

Mathematics Extension 1

Trigonometric functions

Compound angles

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Functions

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\dfrac{d}{a}$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$

$X \sim \mathrm{Bin}(n, p)$

$\mathrm{E}(X) = np$

$\mathrm{Var}(X) = np(1-p)$

Differential calculus

Function	Derivative
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral calculus

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c$$

Vectors

$|\mathbf{u}| = |x\mathbf{i} + y\mathbf{j}| = \sqrt{x^2 + y^2}$

$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = x_1x_2 + y_1y_2$

where $\mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j}$

and $\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j}$

$$\boxed{\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}}$$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n$$

Mathematics Advanced

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Financial mathematics

$$FV = PV(1 + r)^n$$

Sequences and series

$$a_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + a_n)$$

$$a_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r}, \; r \neq 1$$

$$S_\infty = \frac{a}{1-r}, \; |r| < 1$$

Logarithmic and exponential functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Relations

$$(x - a)^2 + (y - b)^2 = r^2$$

Trigonometric functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

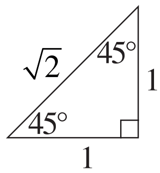
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

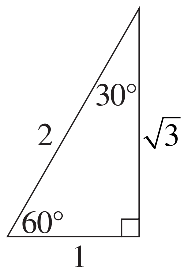
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \; \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \; \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \; \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Differential calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^x$	$\dfrac{dy}{dx} = (\ln a)a^x$
$y = \log_a x$	$\dfrac{dy}{dx} = \dfrac{1}{x\ln a}$

Integral calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int_a^b f(x)\,dx$$

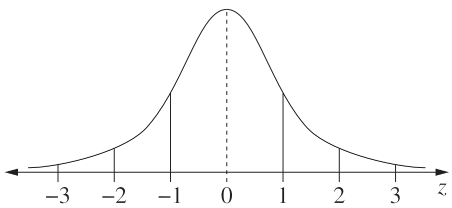
$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Statistical analysis

$$z = \frac{x - \mu}{\sigma}$$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

Discrete random variables

$$\mathrm{E}(X) = \mu$$

$$\mathrm{Var}(X) = \mathrm{E}\left[(X-\mu)^2\right]$$

Continuous random variables

$$P(X \le x) = F(x) = \int_a^x f(t)\,dt$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

$$\boxed{\mathrm{E}(X) = \mu = \int_a^b xf(x)\,dx}$$

$$\boxed{\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \mu^2 = \int_a^b x^2 f(x)\,dx - \mu^2}$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0$$

HSC Mathematics Extension 2 Sample Examination

Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	B
2	A
3	B
4	C
5	D
6	D
7	A
8	B
9	D
10	C

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution	2
• Identifies $\bar{z}$, or equivalent merit	1

Sample answer:

$$z = 5 + i \qquad w = 2 - 4i$$

$$\therefore \bar{z} = 5 - i$$

$$\frac{\bar{z}}{w} = \frac{(5-i)}{(2-4i)} \times \frac{(2+4i)}{(2+4i)}$$

$$= \frac{10 + 20i - 2i + 4}{4 + 16}$$

$$= \frac{14 + 18i}{20}$$

$$= \frac{14}{20} + \frac{18}{20}i$$

$$= \frac{7}{10} + \frac{9}{10}i$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Evaluates one of the three coefficients	2
• Provides a correct expression for the sum of fractions with unknown coefficients, or equivalent merit	1

Sample answer:

Let $$\frac{3x^2-5}{(x-2)(x^2+x+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1}$$

$$= \frac{A(x^2+x+1)+(Bx+C)(x-2)}{(x-2)(x^2+x+1)}$$

$$\therefore 3x^2-5 = A(x^2+x+1)+(Bx+C)(x-2)$$

when $x = 2$

$$3(2)^2-5 = A(2^2+2+1)$$

$$7 = 7A$$

$$A = 1$$

Equating coefficients of x^2

$$3 = A + B$$

$$\therefore B = 2$$

Equating constants:

$$-5 = A - 2C$$

$$2C = 6$$

$$C = 3$$

$$\therefore \frac{3x^2-5}{(x-2)(x^2+x+1)} = \frac{1}{x-2} + \frac{2x+3}{x^2+x+1}$$

Question 11 (c)(i)

Criteria	Marks
• Provides correct solution	2
• Attempts to use the square of $(x + iy)$, or equivalent merit OR • Attempts to use the modulus/argument form of $-i$ OR • Plots $-i$ on an Argand diagram and attempts to locate the square roots	1

Sample answer:

Let $(x+iy)^2 = -i$ where $x, y \in \mathbb{R}$

$\therefore x^2 - y^2 = 0$ and $2xy = -1$

$$\therefore y = \frac{-1}{2x}$$

$$\therefore x^2 - \left(\frac{-1}{2x}\right)^2 = 0$$

$$4x^4 - 1 = 0$$

$$x^4 = \frac{1}{4}$$

$$x^2 = \frac{1}{2} \quad (x \in \mathbb{R})$$

$$\therefore \quad x = \pm\frac{1}{\sqrt{2}}$$

$\therefore$ square roots are $\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$ and $\dfrac{-1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$

Question 11 (c)(ii)

Criteria	Marks
• Provides correct answers	2
• Uses the quadratic formula to obtain $z = -1 \pm \sqrt{-i}$, or equivalent merit	1

Sample answer:

$$z^2 + 2z + 1 + i = 0$$

$$\begin{aligned} \therefore \qquad z &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(1+i)}}{2} \\ &= \frac{-2 \pm 2\sqrt{1-1-i}}{2} \\ &= -1 \pm \sqrt{-i} \end{aligned}$$

From part (i):

$$z = -1 + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \quad \text{or} \quad z = -1 - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

that is,

$$z = \frac{\left(1-\sqrt{2}\right)}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \quad \text{or} \quad z = \frac{-\left(1+\sqrt{2}\right)}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

Question 11 (d)(i)

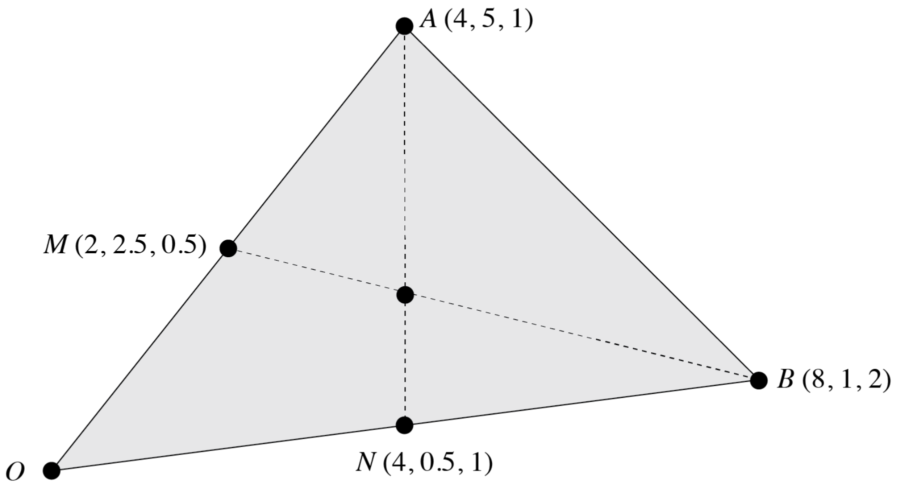
Criteria	Marks
• Provides correct solution	2
• Provides the midpoint of OB, or equivalent merit	1

Sample answer:

Let the midpoint of OB be N, with coordinates (4, 0.5, 1).

$\therefore$ The direction vector of the median through A is $\begin{pmatrix} 0 \\ -4.5 \\ 0 \end{pmatrix}$.

The median from A has equation $\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -4.5 \\ 0 \end{pmatrix}$ for parameter $\mu \in [0,1]$.



Question 11 (d) (ii)

Criteria	Marks
• Provides correct solution	2
• Equates the x components of the vector equations, or equivalent merit	1

Sample answer:

Equating the x-coordinates of the two medians gives $4 + 0\mu = 8 - 6\lambda$.

Hence the point of intersection occurs at $\lambda = \dfrac{2}{3}$.

Question 11 (d)(iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

The centroid divides the median in the ratio of 2:1.

Question 11 (e)(i)

Criteria	Marks
• States the converse	1

Sample answer:

The converse is 'If n is even, then n^2 is even'.

Question 11 (e)(ii)

Criteria	Marks
• Provides correct proof	1

Sample answer:

If n is even then $n = 2k$ for some integer k, then $n^2 = (2k)^2 = 2(2k^2)$ which is also even.

Question 12 (a)

Criteria	Marks
• Provides correct solution	3
• Integrates one of the correct fractions OR • Writes the integrand as the correct sum of fractions and completes the square in a denominator, or equivalent merit	2
• Attempts to write the integrand as a sum of two suitable fractions, or equivalent merit	1

Sample answer:

$$\int \frac{2x+3}{x^2+2x+2}\,dx = \int \frac{2x+2}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

$$= \ln\left(x^2+2x+2\right) + \int \frac{1}{1+(x+1)^2}\,dx + C$$

$$= \ln\left(x^2+2x+2\right) + \tan^{-1}(x+1) + k$$

Question 12 (b)

Criteria	Marks
• Provides correct solution	3
• Shows that the lines do not intersect OR that they are not parallel	2
• Attempts to solve for s or t, or equivalent merit	1

Sample answer:

Skew lines do not intersect and are not parallel.

Solving for s and t in x and y:

From x: $-1 + 2s = 1 + 2t$ (1)

From y: $1 - 2s = -1 - t$ (2)

(1) + (2) $\quad 0 = t$

Substitute into (1)

$$-1 + 2s = 1$$
$$2s = 2$$
$$s = 1$$
$$\therefore s = 1, \quad t = 0$$

Substituting these values into the z equations gives: 1 + 2 and 4 + 0.

The z values are not the same. Therefore, the lines do not intersect.

The direction vector for Line 1 is $\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$ and for Line 2 is $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

$\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \neq k \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ for any $k \in \mathbb{R}$.

Therefore Lines 1 and 2 are not parallel.

Lines 1 and 2 are skew as they do not intersect and are not parallel.

Question 12 (c)(i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$
\begin{aligned}
P(t) &= |\mathbf{u} - t\mathbf{v}|^2 \\
&= (\mathbf{u} - t\mathbf{v}) \cdot (\mathbf{u} - t\mathbf{v}) \\
&= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot (-t\mathbf{v}) + (-t\mathbf{v}) \cdot \mathbf{u} + t^2(\mathbf{v} \cdot \mathbf{v}) \\
&= |\mathbf{u}|^2 - 2t(\mathbf{u} \cdot \mathbf{v}) + t^2|\mathbf{v}|^2
\end{aligned}
$$

Question 12 (c)(ii)

Criteria	Marks
• Provides correct solution	2
• Finds $\frac{dP}{dt}$, or equivalent merit	1

Sample answer:

$$\frac{dP}{dt} = 2t|\mathbf{v}|^2 - 2\mathbf{u} \cdot \mathbf{v} = 0$$

The solution is $t = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}$.

$$\frac{d^2P}{dt^2} = 2|\mathbf{v}|^2 > 0$$

So the stationary point is a minimum.

Question 12 (c)(iii)

Criteria	Marks
• Provides correct solution	3
• Computes the minimum value, or equivalent	2
• Justifies why the minimum value is ≥ 0	1

Sample answer:

Since $P(t)$ is given to be the square of the length of a vector, it is always ≥ 0.

So the minimum value is ≥ 0.

Using the result in part (ii), the minimum value is

$$P\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)^2 |v|^2 - 2\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{u}\cdot\mathbf{v} + |\mathbf{u}|^2$$

$$= -\frac{(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{v}|^2} + |\mathbf{u}|^2$$

Hence $0 \leq -\frac{(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{v}|^2} + |\mathbf{u}|^2$, which simplifies to $(\mathbf{u}\cdot\mathbf{v})^2 \leq |\mathbf{u}|^2|\mathbf{v}|^2$.

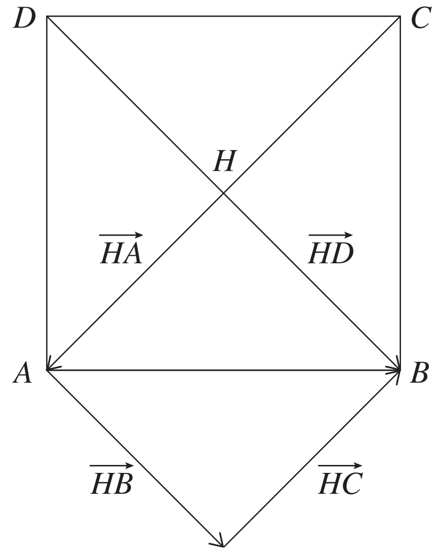
Taking the square root yields the Cauchy–Schwarz inequality $|\mathbf{u}\cdot\mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$.

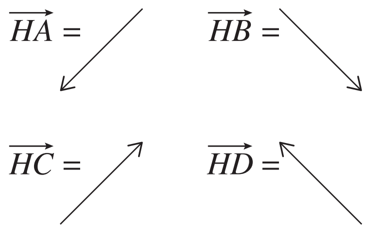
Question 12 (d)(i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Consider the square base





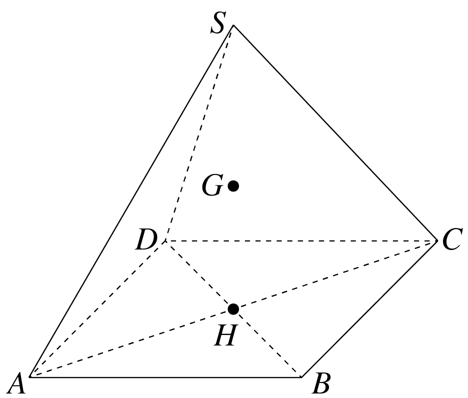
As H is the midpoint of AC and BD, when you add them head to tail, you end up back where you started.

$$\therefore\ \overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD} = \mathbf{0}$$

Question 12 (d)(ii)

Criteria	Marks
• Provides correct solution	2
• Writes $\overrightarrow{GA} = \overrightarrow{GH} + \overrightarrow{HA}$, or equivalent merit	1

Sample answer:



$$\overrightarrow{GA} = \overrightarrow{GH} + \overrightarrow{HA}$$
$$\overrightarrow{GB} = \overrightarrow{GH} + \overrightarrow{HB}$$
$$\overrightarrow{GC} = \overrightarrow{GH} + \overrightarrow{HC}$$
$$\overrightarrow{GD} = \overrightarrow{GH} + \overrightarrow{HD}$$

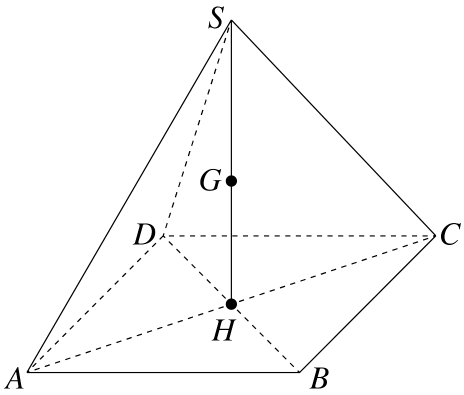
Adding these

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} = 4\overrightarrow{GH} + \overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD}$$
$$-\overrightarrow{GS} = 4\overrightarrow{GH} + \mathbf{0}$$
$$\therefore\ 4\overrightarrow{GH} + \overrightarrow{GS} = \mathbf{0}$$

Question 12 (d)(iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:



From part (ii)

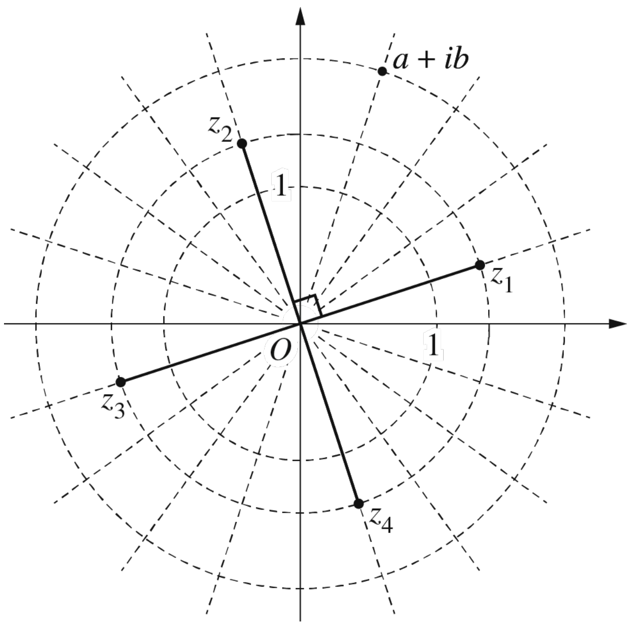
$$
\begin{aligned}
4\overrightarrow{GH} + \overrightarrow{GS} &= \mathbf{0} \\
4\overrightarrow{GH} + \overrightarrow{GH} + \overrightarrow{HS} &= \mathbf{0} \\
5\overrightarrow{GH} + \overrightarrow{HS} &= \mathbf{0} \\
\overrightarrow{HS} &= -5\overrightarrow{GH} \\
&= 5\overrightarrow{HG} \\
\overrightarrow{HG} &= \frac{1}{5}\overrightarrow{HS} \\
\therefore\ \lambda &= \frac{1}{5}
\end{aligned}
$$

Question 13 (a)

Criteria	Marks
• Provides appropriate sketch	2
• Indicates one correct argument OR • Indicates correct modulus	1

Sample answer:

z_1, z_2, z_3 and z_4 represent the fourth roots.



Question 13 (b)(i)

Criteria	Marks
• Provides correct solution	2
• States the amplitude of the motion, or equivalent merit	1

Sample answer:

$\ddot{x} = -4(x - 3)$

$\therefore n^2 = 4, \quad \text{centre is } x = 3$

$n = 2$

$\text{period} = \frac{2\pi}{2} = \pi$

$v^2 = n^2\left(a^2 - (x - c)^2\right)$

when $\quad v = 8, x = 0$

$64 = 4\left(a^2 - (0 - 3)^2\right)$

$16 = a^2 - 9$

$a^2 = 25$

$a = 5$

$\therefore$ The particle oscillates between

$x = 3 + 5 \quad$ and $\quad x = 3 - 5$

$= 8 \qquad\qquad\qquad = -2$

Question 13 (b)(ii)

Criteria	Marks
• Provides correct solution	2
• States the amplitude of the motion, or equivalent merit	1

Sample answer:

Consider $x = a\cos(nt + \alpha) + c$, where

$a = 5, \quad n = 2, \quad c = 3$

$\therefore \; x = 5\cos(2t + \alpha) + 3$

When $\quad t = 0, \quad x = 5.5$

$$5.5 = 5\cos\alpha + 3$$

$$\frac{2.5}{5} = \cos\alpha$$

$$\therefore \qquad \alpha = \cos^{-1}\left(\frac{2.5}{5}\right)$$

$$= \frac{\pi}{3}$$

$$0 = 5\cos\left(2t + \frac{\pi}{3}\right) + 3$$

$$\cos\left(2t + \frac{\pi}{3}\right) = \frac{-3}{5}$$

$$2t + \frac{\pi}{3} = 2.214\ldots\text{radians}$$

$$2t = 1.167\ldots$$

$$t = 0.583\ldots$$

$\therefore$ First value of t when $x = 0$ is $t = 0.58$ seconds (2 decimal places).

Question 13 (c)(i)

Criteria	Marks
• Provides correct solution	2
• States that the maximum height occurs when the vertical component of the velocity is zero, or equivalent merit.	1

Sample answer:

Vertical component of velocity is 0.

$$36.5e^{-0.8t} - 12.5 = 0$$
$$36.5e^{-0.8t} = 12.5$$
$$e^{-0.8t} = 12.5 \div 36.5$$
$$-0.8t = \ln(12.5 \div 36.5)$$
$$t = \ln(12.5 \div 36.5) \div (-0.8)$$
$$= 1.339\,\text{s (to 3 decimal places)}$$

Question 13 (c)(ii)

Criteria	Marks
• Provides correct solution	3
• Calculates the vertical height, B, or equivalent merit	2
• Calculates the horizontal displacement, A, or equivalent merit	1

Sample answer:

Horizontal velocity is $10e^{-0.8t}$

$$\therefore x = \int 10e^{-0.8t}\,dt$$

$$x = \frac{10}{-0.8}e^{-0.8t} + C_1$$

$$x = -12.5e^{-0.8t} + C_1$$

When $t = 0$, $x = 0$

$$\therefore 0 = -12.5e^{0} + C_1$$

$$\therefore C = 12.5$$

$$\therefore x = -12.5e^{-0.8t} + 12.5$$

When $t = 1.339$, $x = A$

$$\therefore A = -12.5e^{-0.8\times 1.339} + 12.5$$

$$\therefore A = 8.217\text{m (to 3 decimal places)}$$

Vertical velocity is $36.5e^{-0.8t} - 12.5$

$$\therefore y = \int \left(36.5e^{-0.8t} - 12.5\right)dt$$

$$y = \frac{36.5}{-0.8}e^{-0.8t} - 12.5t + C_2$$

$$y = -45.625e^{-0.8t} - 12.5t + C_2$$

When $t = 0$, $y = 0$

$$\therefore 0 = -45.625 + C_2$$

$$\therefore C_2 = 45.625$$

$$\therefore y = -45.625e^{-0.8t} - 12.5t + 45.625$$

When $t = 1.339$, $y = B$

$$\therefore B = -45.625e^{-0.8\times 1.339} - 12.5\times 1.339 + 45.625$$

$$\therefore B = 13.257\text{m (to 3 decimal places)}$$

Question 13 (c)(iii)

Criteria	Marks
• Provides the correct solution	1

Sample answer:

At the top of the trajectory, vertical component of velocity = 0

$$\begin{aligned} \text{speed} &= \text{horizontal component of velocity} \\ &= 10e^{-0.8t} \\ \text{when } t = 1.339, &= 10e^{-0.8\times 1.339} \\ &= 3.426 \text{ ms}^{-2} (\text{to 3 decimal places}) \end{aligned}$$

Question 13 (c) (iv)

Criteria	Marks
• Provides correct solution	2
• Differentiates $\mathbf{v}(t)$ to get the acceleration vector, or equivalent merit	1

Sample answer:

Force = mass × acceleration

$$\mathbf{v}(t) = 10e^{-0.8t}\mathbf{i} + \left(36.5e^{-0.8t} - 12.5\right)\mathbf{j}$$

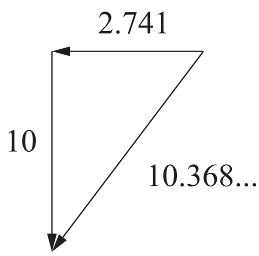
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -8e^{-0.8t}\mathbf{i} - 29.2e^{-0.8t}\mathbf{j}$$

when $t = 1.339$

$$\begin{aligned} \mathbf{a} &= -8e^{-0.8\times 1.339}\mathbf{i} - 29.2e^{-0.8\times 1.339}\mathbf{j} \\ &= -2.741\mathbf{i} - 10\mathbf{j} \end{aligned}$$

Magnitude of acceleration

$$\begin{aligned} &= \sqrt{(2.741)^2 + (10)^2} \\ &= 10.36885\ldots \text{ms}^{-2} \end{aligned}$$



∴ Magnitude of force

$$\begin{aligned} &= 8 \times 10.36885\ldots \\ &= 82.951 \text{ newtons } (\text{to 3 decimal places}) \end{aligned}$$

Question 14 (a)

Criteria	Marks
• Provides correct solution	4
• Applies partial fraction to the correct integral, or equivalent merit	3
• Completes t-substitution, including the limits of integration, or equivalent merit	2
• Attempts to use a t-substitution, or equivalent merit	1

Sample answer:

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\,dx \qquad \text{let } t = \tan\frac{x}{2} \qquad \therefore \cos x = \frac{1-t^2}{1+t^2}$$

when $x = 0, \quad t = 0$

$x = \frac{\pi}{2}, \quad t = 1$

$$dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left(1+t^2\right)dx$$

$$\therefore dx = \frac{2}{1+t^2}\,dt$$

$$\therefore \text{Integral} = \int_0^1 \frac{1}{3+\frac{5\left(1-t^2\right)}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{3+3t^2+5-5t^2}\,dt$$

$$= \int_0^1 \frac{2}{8-2t^2}\,dt$$

$$= \int_0^1 \frac{1}{4-t^2}\,dt$$

$$= \int_0^1 \frac{1}{(2-t)(2+t)}\,dt$$

$$= \frac{1}{4}\int_0^1 \frac{1}{2-t} + \frac{1}{2+t}\,dt$$

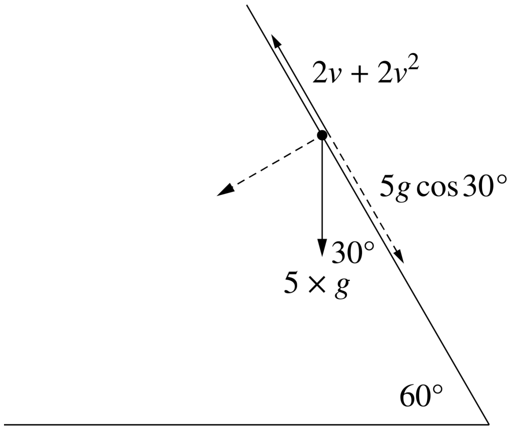
$$= \frac{1}{4}\left[\ln\left|\frac{2+t}{2-t}\right|\right]_0^1$$

$$= \frac{1}{4}(\ln 3 - \ln 1) \; = \frac{1}{4}\ln 3$$

Question 14 (b)(i)

Criteria	Marks
• Provides correct solution	2
• Finds appropriate component of force due to gravity OR • Provides appropriate diagram, or equivalent merit	1

Sample answer:



$$\text{Resultant force down slope } = 5g\cos 30° - 2v - 2v^2$$
$$= \frac{5\sqrt{3}}{2}g - 2v - 2v^2 \text{ newtons}$$

Question 14 (b)(ii)

Criteria	Marks
• Provides correct solution	2
• States that the resultant force is **0**, or equivalent merit	1

Sample answer:

Constant speed $\Rightarrow$ Resultant force = 0

$$\therefore \frac{5\sqrt{3}}{2}g = 2v + 2v^2$$

$$g = 10 \quad \therefore 25\sqrt{3} = 2v + 2v^2$$

$$2v^2 + 2v - 25\sqrt{3} = 0$$

$$\therefore v = \frac{-2 \pm \sqrt{4 + 4(2)\left(25\sqrt{3}\right)}}{4}$$

$$= 4.1798\ldots \text{ or } -5.1798\ldots$$

$\therefore$ speed $= 4.1798\ldots$ as moving down the slope

$\simeq 4.2 \text{ m s}^{-1}$ (1 decimal place)

Question 14 (c)(i)

Criteria	Marks
• Provides correct solution	3
• Expands using binomial theorem • Expands using De Moivre's theorem	2
• Expands using binomial theorem OR • Expands using De Moivre's theorem	1

Sample answer:

Using De Moivre:

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Using binomial expansion

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^5 = &\ (\cos\theta)^5 + 5(\cos\theta)^4(i\sin\theta) \\
&+10(\cos\theta)^3(i\sin\theta)^2 \\
&+10(\cos\theta)^2(i\sin\theta)^3 \\
&+5(\cos\theta)(i\sin\theta)^4 \\
&+(i\sin\theta)^5 \\
= &\ \cos^5\theta + 5\cos^4\theta\sin\theta i \\
&-10\cos^3\theta\sin^2\theta \\
&-10\cos^2\theta\sin^3\theta i \\
&+5\cos\theta\sin^4\theta \\
&+\sin^5\theta i
\end{aligned}
$$

Equating real parts:

$$
\begin{aligned}
\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\
&= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-\cos^2\theta\right)^2 \\
&= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta\left(1-2\cos^2\theta+\cos^4\theta\right) \\
&= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}
$$

Question 14 (c)(ii)

Criteria	Marks
• Provides correct solution	3
• Finds the solutions to the degree 5 polynomial, or equivalent merit	2
• Recognise that $\cos 5\theta = 0$, or equivalent merit	1

Sample answer:

Suppose $\cos\theta + i\sin\theta$ is the 5th root of i.

$i = (\cos\theta + i\sin\theta)^5 = \cos(5\theta) + i\sin(5\theta)$

Equating the real parts, $\cos(5\theta) = 0$

By part (i), $\cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right) = 0$

$\therefore \cos\theta = 0 \quad \text{or} \quad 16\cos^4\theta - 20\cos^2\theta + 5 = 0$

By the quadratic formula, $\cos^2\theta = \dfrac{20 \pm \sqrt{400 - 4\times 16\times 5}}{32} = \dfrac{5\pm\sqrt{5}}{8}$.

Hence $\cos\theta = \pm\sqrt{\dfrac{5\pm\sqrt{5}}{8}}$.

So, the real parts of the 5th roots of i are 0 and $\pm\sqrt{\dfrac{5\pm\sqrt{5}}{8}}$.

Question 15 (a)

Criteria	Marks
• Provides correct solution	3
• Obtains correct primitive OR correct definite integral in terms of u, or equivalent merit	2
• Attempts to use a suitable substitution	1

Sample answer:

$$I = \int_{\sqrt{10}}^{\sqrt{13}} x^3\sqrt{x^2-9}\,dx$$

$$= \int_{\sqrt{10}}^{\sqrt{13}} x^2\left(x^2-9\right)\frac{x}{\sqrt{x^2-9}}\,dx$$

$$= \int_1^2 \left(u^2+9\right)u^2\,du$$

$$= \int_1^2 u^4+9u^2\,du$$

$$= \left[\frac{u^5}{5}+3u^3\right]_1^2$$

$$= \frac{2^5}{5}+3\times 2^3-\left(\frac{1}{5}+3\right)$$

$$= \frac{32}{5}-\frac{1}{5}+24-3$$

$$= \frac{31}{5}+21$$

$$= \frac{136}{5}$$

Let $u = \sqrt{x^2-9}$

$$du = \frac{x}{\sqrt{x^2-9}}\,dx$$

and $u^2 = x^2-9$

When $x = \sqrt{10}$, $u = \sqrt{10-9} = 1$

When $x = \sqrt{13}$, $u = \sqrt{13-9} = 2$

Question 15 (b)(i)

Criteria	Marks
• Provides correct solution	3
• Integrates to obtain an expression for t in terms of v, or equivalent merit	2
• Provides an expression for the resultant force, or equivalent merit	1

Sample answer:

$$\text{Force} = -mg - kv^2$$

$$ma = -mg - kv^2$$

$$a = -g - \frac{kv^2}{m}$$

$$a = -g - kv^2 \qquad \text{given } m = 1$$

$$\frac{dv}{dt} = -g - kv^2 \qquad \text{need } t \text{ in terms of } v\text{, so use } a = \frac{dv}{dt}$$

$$\frac{dt}{dv} = -\frac{1}{g + kv^2}$$

$$t = -\frac{1}{\sqrt{gk}}\tan^{-1}\left(v\sqrt{\frac{k}{g}}\right) + c$$

When $t = 0$, $v = u$

$$0 = -\frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) + c$$

$$c = \frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

$$\therefore \quad t = \frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) - \frac{1}{\sqrt{gk}}\tan^{-1}\left(v\sqrt{\frac{k}{g}}\right)$$

Max height, $v = 0 \therefore t = \frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$ as required.

Question 15 (b)(ii)

Criteria	Marks
• Provides correct solution	3
• Integrates to obtain an expression for x in terms of v, or equivalent merit	2
• Provides an integral expression for x in terms of v, or equivalent merit	1

Sample answer:

$$v\frac{dv}{dx} = -g - kv^2$$

$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$x = -\frac{1}{2k}\ln\left(g + kv^2\right) + c$$

When $x = 0,\ v = u$

$$0 = -\frac{1}{2k}\ln\left(g + ku^2\right) + c$$

$$c = \frac{1}{2k}\ln\left(g + ku^2\right)$$

$$\therefore\ x = \frac{1}{2k}\left(\ln\left(g + ku^2\right) - \ln\left(g + kv^2\right)\right) = \frac{1}{2k}\ln\frac{g + ku^2}{g + kv^2}$$

Maximum height, sub in $v = 0$

$\therefore$ maximum height $x = \dfrac{1}{2k}\ln\left(\dfrac{g + ku^2}{g}\right)$

Question 15 (c)(i)

Criteria	Marks
• Provides correct solution	3
• Correctly applies integration by parts, or equivalent merit	2
• Attempts to apply integration by parts or equivalent merit	1

Sample answer:

$$I_n = \int_{-3}^{0} x^n \sqrt{x+3}\, dx \qquad n = 0, 1, 2\ldots$$

Let $\quad u = x^n \qquad v' = (x+3)^{\frac{1}{2}}$

$$u' = nx^{n-1} \qquad v = \frac{2(x+3)^{\frac{3}{2}}}{3}$$

$$I_n = \left[\frac{2x^n(x+3)^{\frac{3}{2}}}{3}\right]_{-3}^{0} - \int_{-3}^{0} \frac{2nx^{n-1}(x+3)^{\frac{3}{2}}}{3}\, dx$$

$$I_n = (0-0) - \frac{2n}{3}\int_{-3}^{0} x^{n-1}(x+3)\sqrt{x+3}\, dx$$

$$I_n = -\frac{2n}{3}\int_{-3}^{0} \left(x^n\sqrt{x+3} + 3x^{n-1}\sqrt{x+3}\right) dx$$

$$I_n = -\frac{2n}{3}\int_{-3}^{0} x^n\sqrt{x+3}\, dx - 2n\int_{-3}^{0} x^{n-1}\sqrt{x+3}\, dx$$

$$I_n = -\frac{2n}{3} I_n - 2nI_{n-1}$$

$$I_n\left(1 + \frac{2n}{3}\right) = -2nI_{n-1}$$

$$I_n\left(\frac{3+2n}{3}\right) = -2nI_{n-1}$$

$$I_n = \frac{-6n}{3+2n} I_{n-1}$$

Question 15 (c)(ii)

Criteria	Marks
• Provides correct solution	2
• Finds I_2 in terms of I_0, or equivalent merit	1

Sample answer:

$$I_2 = \frac{-12}{7}\,I_1$$

$$= \frac{-12}{7} \times \frac{-6}{5}\,I_0$$

$$I_0 = \int_{-3}^{0} \sqrt{x+3}\,dx$$

$$= \left[\frac{2(x+3)^{\frac{3}{2}}}{3}\right]_{-3}^{0}$$

$$= \frac{2}{3}(3)^{\frac{3}{2}} - 0$$

$$= \frac{2 \times 3\sqrt{3}}{3}$$

$$= 2\sqrt{3}$$

$$\therefore\ I_2 = \frac{72}{35} \times 2\sqrt{3}$$

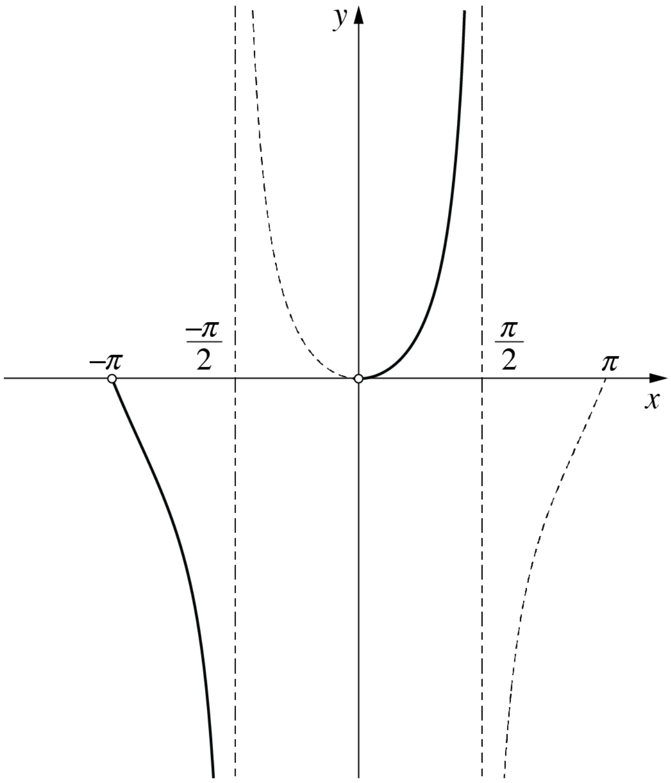
$$= \frac{144\sqrt{3}}{35}$$

Question 16 (a)

Criteria	Marks
• Provides correct sketch	3
• Provides a sketch $y = x\tan x$,or equivalent merit	2
• Attempts to find the Cartesian equation using $\mathrm{Re}(z)$ and $\mathrm{Arg}(z)$	1

Sample answer:

Since $\mathrm{Re}(z) = x$ and $\tan\left(\mathrm{Arg}(z)\right) = \frac{y}{x}(x \neq 0)$, the solution to $\mathrm{Re}(z) = \mathrm{Arg}(z)$ satisfies $y = x\tan(x)$.



In quadrant 1, the real part and argument are both positive.

In quadrant 2, the real part is negative but the argument is positive, so this is excluded.

In quadrant 3, the real part and argument are both negative.

In quadrant 4, the real part is positive but the argument is negative, so this is excluded

The origin is excluded because the argument is undefined at the origin. The complex number $-\pi + 0i$ is excluded because is has a real part $-\pi$ but the argument is π.

The set of complex numbers satisfying $\mathrm{Re}(z) = \mathrm{Arg}(z)$ is shown by the solid lines in the sketch above.

Question 16 (b)

Criteria	Marks
• Provides correct solution	4
• Provides an equation in terms of λ , or equivalent merit	3
• Provides equations for the sphere and line, or equivalent merit	2
• Provides the equation of the sphere, or equivalent merit	1

Sample answer:

The sphere centred at $C(6, -3, 2)$ with radius $r = 3$

$$(x-6)^2 + (y+3)^2 + (z-2)^2 = 3^2$$
$$= 9$$

The line AB has direction

$$\overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 5-3 \\ -1-(-1) \\ -5-6 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -11 \end{bmatrix}$$

so the line AB has parametric equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$$
$$= \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 0 \\ -11 \end{bmatrix}$$
$$= \begin{bmatrix} 3+2\lambda \\ -1 \\ 6-11\lambda \end{bmatrix}$$

Substitute into the equation for the sphere

$$(3 + 2\lambda - 6)^2 + (-1 + 3)^2 + (6 - 11\lambda - 2)^2 = 9$$
$$(2\lambda - 3)^2 + 4 + (4 - 11\lambda)^2 = 9$$

Expand each term and simplify

$$9 - 12\lambda + 4\lambda^2 + 4 + 16 - 88\lambda + 121\lambda^2 = 9$$
$$125\lambda^2 - 100\lambda + 20 = 0$$
$$5\left(25\lambda^2 - 20\lambda + 4\right) = 5(5\lambda - 2)^2 = 0$$

Hence there is exactly one λ for which there is a common point on the line and the sphere, so the line is a tangent to the sphere.

Question 16 (c)(i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Since $a_1 b_1 = 2$ and $a_{n+1} b_{n+1} = a_{n+1} \times \dfrac{2}{a_{n+1}} = 2$, each rectangle has area 2.

Question 16 (c)(ii)

Criteria	Marks
• Provides correct solution	2
• Attempts to use the relationship between the arithmetic mean and the geometric mean	1

Sample answer:

Since $a_1 \geq \sqrt{2}$ and $a_{n+1} = \dfrac{a_n + b_n}{2} \geq \sqrt{a_n b_n} = \sqrt{2}$ by the relationship between the arithmetic mean and the geometric mean, we have $a_{n+1} \geq \sqrt{2}$ which implies $a_n \geq \sqrt{2}$.

Question 16 (c)(iii)

Criteria	Marks
• Provides correct solution	4
• Shows $b_k - \sqrt{2} \le 0$, or equivalent merit	3
• Attempts to use inductive hypothesis	2
• Proves base case, or equivalent merit	1

Sample answer:

Base case is when $n = 1$. Base case is true because $a_1 - \sqrt{2} = 2 - \sqrt{2} \le \frac{1}{2^{1-1}}\left(2 - \sqrt{2}\right)$.

Assume true for $n = k$, ie $a_k - \sqrt{2} \le \frac{1}{2^{k-1}}\left(2 - \sqrt{2}\right)$ for some $k \ge 1$.

Required to prove true for $n = k + 1$, ie $a_{k+1} - \sqrt{2} \le \frac{1}{2^k}\left(2 - \sqrt{2}\right)$.

By definition, $a_{k+1} = \frac{1}{2}\left(a_k + b_k\right)$.

So, $a_{k+1} - \sqrt{2} = \frac{1}{2}\left(a_k - \sqrt{2}\right) + \frac{1}{2}\left(b_k - \sqrt{2}\right) \le \frac{1}{2}\left(\frac{1}{2^{k-1}}\left(2 - \sqrt{2}\right)\right) + \frac{1}{2}\left(b_k - \sqrt{2}\right)$ by induction assumption.

Since $a_k \ge \sqrt{2}$, we have $b_k \le \sqrt{2}$ and $b_k - \sqrt{2} \le 0$.

So $a_{k+1} - \sqrt{2} \le \frac{1}{2^k}\left(2 - \sqrt{2}\right) + \frac{1}{2}\left(b_k - \sqrt{2}\right) \le \frac{1}{2^k}\left(2 - \sqrt{2}\right)$.

This proves the statement for $n = k + 1$. Hence it is true for all $n \ge 1$ by mathematical induction.

Question 16 (c)(iv)

Criteria	Marks
• Provides correct solution	2
• Uses previous parts to set-up squeeze theorem	1

Sample answer:

Parts (ii) and (iii) combine into $0 \le a_n - \sqrt{2} \le \frac{1}{2^{n-1}}\left(2 - \sqrt{2}\right)$.

As n approaches infinity, $\frac{1}{2^{n-1}}$ approaches zero. Hence, $a_n - \sqrt{2}$ approaches zero by the squeeze theorem.

So a_n approaches $\sqrt{2}$. Since the rectangles have an area of 2, b_n approaches $\sqrt{2}$ also. Which is to say that the rectangles approach a square.

HSC Mathematics Extension 2 Sample Examination

Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes	Targeted performance bands
1	1	Further work with vectors – Vectors and geometry	ME2-12-02	E2–E3
2	1	Further Integration	ME2-12-04	E2–E3
3	1	Further work with vectors – Vector equations of lines and curves	ME2-12-02	E2–E3
4	1	The nature of proof – The language and notation of proof	ME2-12-01	E2–E3
5	1	The nature of proof – The language and notation of proof	ME2-12-01	E2–E3
6	1	Introduction to complex numbers – Describing lines, curves and regions	ME2-12-03	E3–E4
7	1	Introduction to complex numbers – Solving equations with complex numbers	ME2-12-03	E3–E4
8	1	Further work with vectors – Vector equations of lines and curves	ME2-12-02	E3–E4
9	1	The nature of proof – Illustrations of proofs	ME2-12-01	E3–E4
10	1	Applications of calculus to mechanics – Vertical resisted motion	ME2-12-05	E3–E4

Section II

Question	Marks	Content	Syllabus outcomes	Targeted performance bands
11(a)	2	Introduction to complex numbers – Arithmetic of complex numbers	ME2-12-03	E2–E3
11(b)	3	Further integration	ME2-12-04	E2–E3
11(c)(i)	2	Introduction to complex numbers – Arithmetic of complex numbers	ME2-12-03	E2–E3
11(c)(ii)	2	Introduction to complex numbers – Solving equations with complex numbers	ME2-12-03	E2–E3
11(d)(i)	2	Further work with vectors – Vectors and geometry	ME2-12-02	E2–E3
11(d)(ii)	2	Further work with vectors – Vector equations of lines and curves	ME2-12-02	E2–E3
11(d)(iii)	1	Further work with vectors – Vector equations of lines and curves	ME2-12-02	E3–E4
11(e)(i)	1	Nature of proof – The language and notation of proof	ME2-12-01	E2–E3
11(e)(ii)	1	Nature of proof – Illustrations of proofs	ME2-12-01	E2–E3
12(a)	3	Further integration	ME2-12-04	E2–E3

Question	Marks	Content	Syllabus outcomes	Targeted performance bands
12(b)	3	Further work with vectors – Vector equations of lines and curves	ME2-12-02	E2–E4
12(c)(i)	1	Further work with vectors – Vectors and geometry	ME2-12-02	E2–E3
12(c)(ii)	2	Further work with vectors – Vectors and geometry	ME2-12-02	E2–E3
12(c)(iii)	3	Further work with vectors – Vectors and geometry	ME2-12-02	E3–E4
12(d)(i)	1	Further work with vectors – Vectors and geometry	ME2-12-02	E2–E3
12(d)(ii)	2	Further work with vectors – Vectors and geometry	ME2-12-02	E3–E4
12(d)(iii)	1	Further work with vectors – Vectors and geometry	ME2-12-02	E3–E4
13(a)	2	Introduction to complex numbers – Powers and roots of complex numbers	ME2-12-03	E2–E3
13(b)(i)	2	Applications of calculus to mechanics – Simple harmonic motion	ME2-12-05	E2–E3
13(b)(ii)	2	Applications of calculus to mechanics – Simple harmonic motion	ME2-12-05	E3–E4
13(c)(i)	2	Applications of calculus to mechanics – Projectiles and resisted motion	ME2-12-05	E2–E3
13(c)(ii)	3	Applications of calculus to mechanics – Projectiles and resisted motion	ME2-12-05	E2–E4
13(c)(iii)	1	Applications of calculus to mechanics – Projectiles and resisted motion	ME2-12-05	E2–E3
13(c)(iv)	2	Applications of calculus to mechanics – Projectiles and resisted motion	ME2-12-05	E3–E4
14(a)	4	Further integration	ME2-12-04	E2–E4
14(b)(i)	2	Applications of calculus to mechanics – Modelling motion without resistance	ME2-12-05	E3–E4
14(b)(ii)	2	Applications of calculus to mechanics – Modelling motion without resistance	ME2-12-05	E2–E3
14(c)(i)	3	Introduction to complex numbers – Powers and roots of complex numbers	ME2-12-03	E2–E3
14(c)(ii)	3	Introduction to complex numbers – Powers and roots of complex numbers	ME2-12-03	E3–E4
15(a)	3	Further integration	ME2-12-04	E2–E4
15(b)(i)	3	Applications of calculus to mechanics – Vertical resisted motion	ME2-12-05	E2–E4
15(b)(ii)	3	Applications of calculus to mechanics – Vertical resisted motion	ME2-12-05	E2–E4
15(c)(i)	3	Further integration	ME2-12-04	E2–E4
15(c)(ii)	2	Further integration	ME2-12-04	E2–E4
16(a)	3	Introduction to complex numbers – Geometric representation of complex numbers	ME2-12-03	E3–E4
16(b)	4	Further work with vectors – Vector equations of lines and curves	ME2-12-02	E2–E4
16(c)(i)	1	Nature of proof – Further proof by mathematical induction	ME2-12-01	E2–E3

Question	Marks	Content	Syllabus outcomes	Targeted performance bands
16(c)(ii)	2	Nature of proof – Proof of inequalities	ME2-12-01	E2–E3
16(c)(iii)	4	Nature of proof – Further proof by mathematical induction	ME2-12-01	E2–E4
16(c)(iv)	2	Nature of proof – Proof of inequalities	ME2-12-01	E3–E4